QCD on the lattice - an introduction
Lecture 4

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Hadron spectroscopy (1)

- **Masses** of (colourless) QCD bound-states can be computed by measuring **two-point functions**. The Euclidean two-point function is

$$C(t) = \langle 0|\Phi(t)\Phi^\dagger(0)|0\rangle$$

- The time-dependence of the operator, Φ is given by $\Phi(t) = e^{Ht}\Phi e^{-Ht}$, so

$$C(t) = \langle \Phi|e^{-Ht}|\Phi^\dagger\rangle$$

inserting a complete set of energy eigenstates gives

$$C(t) = \sum_{k=0}^{\infty} \langle \Phi|e^{-Ht}|k\rangle\langle k|\Phi^\dagger\rangle = \sum_{k=0}^{\infty} |\langle \Phi|k\rangle|^2 e^{-E_k t}$$

- Then $\lim_{t\to\infty} C(t) = Ze^{-E_0 t}$
- If the large-time exponential fall-off of the correlation function can be observed, the energy of the state can be measured.

Hadron spectroscopy (2)

- The energies of **excited states** can be computed reliably too.
- Tracking sub-leading exponential fall-off works sometimes but a more efficient method is to use a matrix of correlators. With a set of N operators $\{\Phi_1, \Phi_2, \dots\}$ (with the same quantum numbers), compute all elements of

$$C_{ij}(t) = \langle 0|\Phi_i(t)\Phi_j^\dagger(0)|0\rangle$$

- Now solve the generalised eigenvalue problem

$$C(t_1)v = \lambda C(t_0)v$$

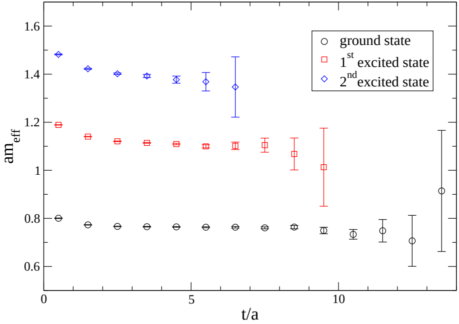
  for different t_0 and t_1.
- The method constructs an optimal linear combination to form a ground-state, and then constructs a set of operators that are orthogonal to it.
- The second eigenvector can not have overlap with the ground-state at large t, and will fall to the first excited energy.

Hadron spectroscopy (3)

- Lattice practitioners like to show this in an **"effective mass plot"**. The effective mass is

$$m_{\rm eff}(t) = -\frac{1}{a}\log\frac{C(t+a)}{C(t)}$$

and for times large enough such that C is dominated by the ground-state, the effective mass should become independent of time; a "plateau".



- Radial (?) excitations of a "static-light" meson.

Spin on the lattice

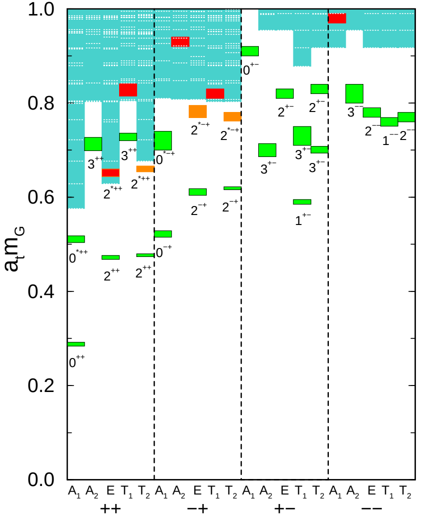
- Eigenstates of the hamiltonian simultaneously form irreducible representations of $SO(3)$, the rotation group. Spin is a good quantum number.
- The lattice hamiltonian does not have $SO(3)$ symmetry. It is symmetric under the discrete sub-group of **rotations of the cube**, O_h. This group has 48 elements (once parity is included) and ten irreducible representations.
- The eigenstates of the lattice hamiltonian therefore have a good "quantum letter"; $A_1^{u,g}, A_2^{u,g}, E^{u,g}, T_1^{u,g}, T_2^{u,g}$
- Can we deduce the continuum spin of a state? With some caveats, yes.
- A pattern of degeneracies must be found and matched against the representations of O_h subduced from $SO(3)$.

Spin on the lattice (2)

J	A_1	A_2	E	T_1	T_2
0	1	—	—	—	—
1	—	—	—	1	—
2	—	—	1	—	1
3	—	1	—	1	1
4	1	—	1	1	1
5	—	—	1	2	1
6	1	1	1	1	2
			⋮		

Example

The Yang-Mills glueball spectrum

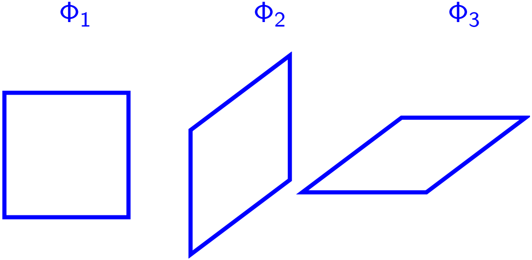


Creation operators: glueballs

- To measure the correlation functions, we need to measure appropriate creation operators on our ensemble.
- The operators should be functions of the fields on a time-slice and transform irreducibly according to an irrep of O_h (as well as isospin, charge conjugation etc.)
- First example: the glueball. An appropriate operator would be a gauge invariant function of the gluons alone: a closed loop trace.
- Link smearing greatly improved ground-state overlap.
- Apply smoothing filters to the links to extract just slowly varying modes that then have better overlap with the lowest states.

Creation operators: glueballs

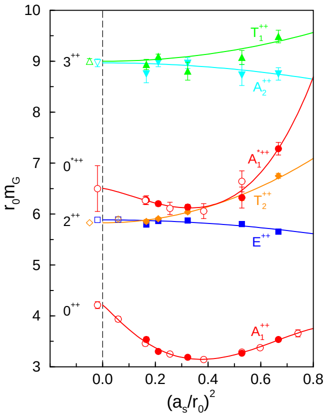
- What do operators that transform irreducibly under O_h look like?

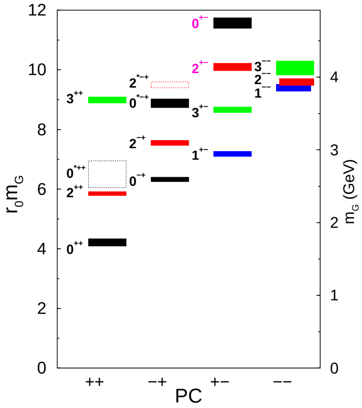


- Can make three operators by taking linear combinations of these loops.
- They form two irreducible representations (A_1^g and E_g).

$$
\begin{array}{ccccccc}
\Phi_{A_1^g} & = & \Phi_1 & + & \Phi_2 & + & \Phi_3 \\
\hline
\Phi_{E^g}^{(1)} & = & \Phi_1 & - & \Phi_2 & & \\
\Phi_{E^g}^{(2)} & = & \frac{1}{\sqrt{3}}(\Phi_1 & + & \Phi_2 & - & 2\Phi_3)
\end{array}
$$

- After running simulations at more than one lattice spacing, a **continuum extrapolation** ($a \rightarrow 0$) can be attempted.
- The expansion of the action can suggest the appropriate choice of extrapolating function.





Isovector meson correlation functions

- To create a meson, we need to build functions that couple to quarks.
- In the simplest model, a meson would be created by a quark bilinear, so the appropriate gauge invariant creation operator (for isospin $I = 1$) would be

$$\Phi_{\mathrm{meson}}(t) = \sum_{x} \bar{u}(\underline{x}, t)\Gamma U_{\mathcal{C}}(\underline{x}, \underline{y}; t) d(\underline{y}, t)$$

  where Γ is some appropriate Dirac structure, and $U_{\mathcal{C}}$ a product of (smeared) link variables.
- As before, appropriate operators that transform irreducibly under the lattice rotation group O_h are needed.
- The complication here is that we do not have direct access to the fermion integration variables in the computer.
- As with updating algorithms, the observation that the quark action is bilinear saves us:

$$\langle \psi_a^{\alpha}(\underline{x}, t)\bar{\psi}_b^{\beta}(\underline{y}, t')\rangle = [M^{-1}]_{ab}^{\alpha,\beta}(x, t; y, t')$$

Isovector meson correlation functions (2)

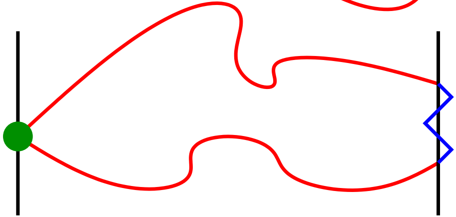
- Now the elementary component in the correlation function is

$$\langle 0|\Phi(t)\Phi^\dagger(0)|0\rangle = \\ \langle \mathrm{Tr}\; M^{-1}(\underline{z},0;\underline{x},t)\Gamma U_{\mathcal{C}}(\underline{x},\underline{y},t)M^{-1}(\underline{y},t;\underline{w},0)\Gamma^\dagger U_{\mathcal{C}'}(\underline{w},\underline{z},0)\rangle$$

- In general, this is still expensive to compute, since it requires knowing many entries in the inverse of the fermion operator, M.
- If the choice of operator at the source is restricted and no momentum projection is made, only the bilinear at (eg) the origin on time-slice 0 is needed.
- Quark propagation from a single site to any other site is computed by solving $M\psi = e_0^{a,\alpha}$ where e_0 are the 12 vectors that only has non-zero components at the origin.
- Getting away from this restriction by estimating "all-to-all" propagators is an active research topic.

Isovector meson correlation functions (3)

The most general operator.



A restricted correlation function accessible to one point-to-all computation.

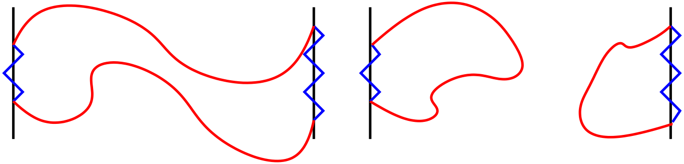
Isoscalar meson correlation functions (1)

- If we are interested in measuring isoscalar meson masses, extra diagrams must be evaluated, since four-quark diagrams become relevant. The Wick contraction yields extra terms, since

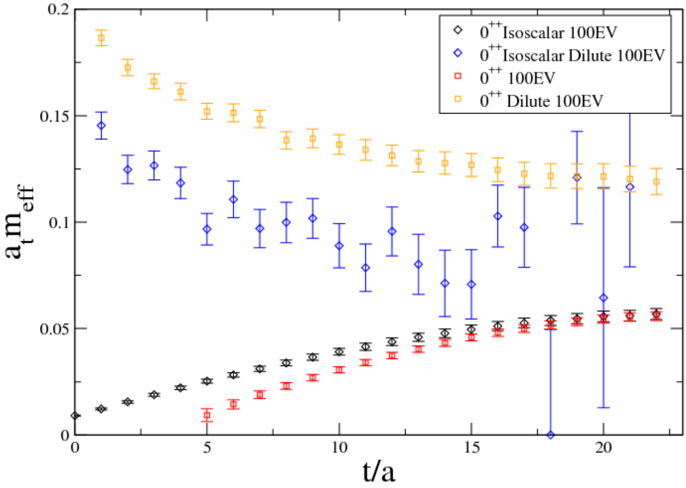
$$\langle \psi_i \bar{\psi}_j \psi_k \bar{\psi}_l \rangle = M^{-1}_{ij} M^{-1}_{kl} - M^{-1}_{il} M^{-1}_{jk}$$

- Now

$$\langle 0|\Phi_{I=0}(t)\Phi^\dagger_{I=0}(0)|0\rangle =$$
$$\langle 0|\Phi_{I=1}(t)\Phi^\dagger_{I=1}(0)|0\rangle - \langle 0|\mathrm{Tr}\ M^{-1}\Gamma U_{\mathcal{C}}(t)\mathrm{Tr}\ M^{-1}\Gamma U_{\mathcal{C}}(0)|0\rangle$$



Isoscalar meson correlation functions (1)



Summary

- Euclidean metric is useful for spectroscopy: isolate ground-state by looking at large-time separation of correlation function
- Analysis of matrices of correlation functions gives a robust way of extracting excited states
- Discretising space-time breaks the rotation symmetry. Lattice energy eigenstates are irreducible representations of the discrete cubic point group (or the little group at finite momentum).
- Appropriate gauge invariant creation operators for many different states can be defined and their correlation functions measured by Monte Carlo.
- For high precision, a continuum extrapolation of data is required.
- Isovector mesons can be probed using point-to-all quark propagation
- Isoscalar mesons need all-to-all methods.