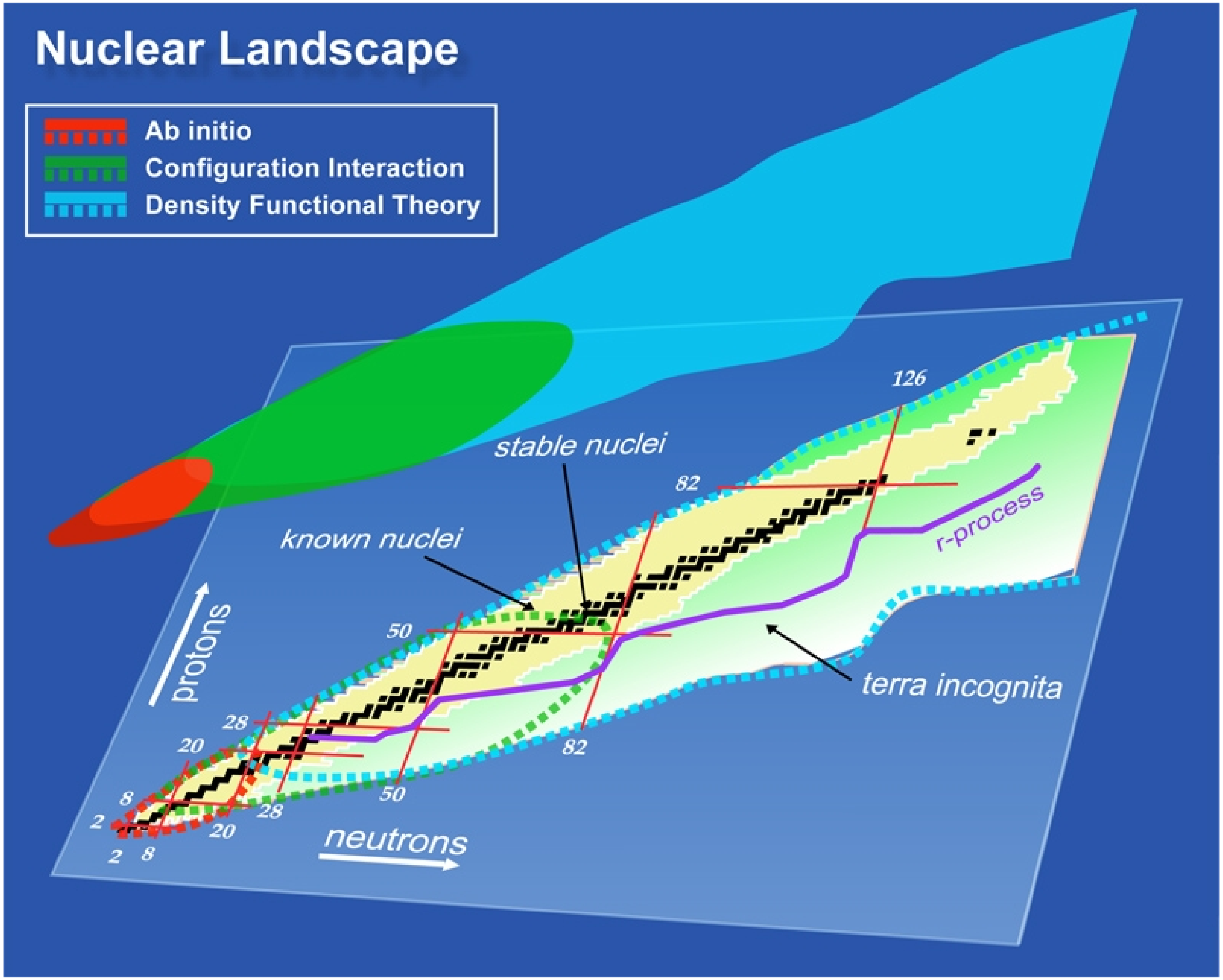
The Nuclear Many-Body Problem
Lecture 3

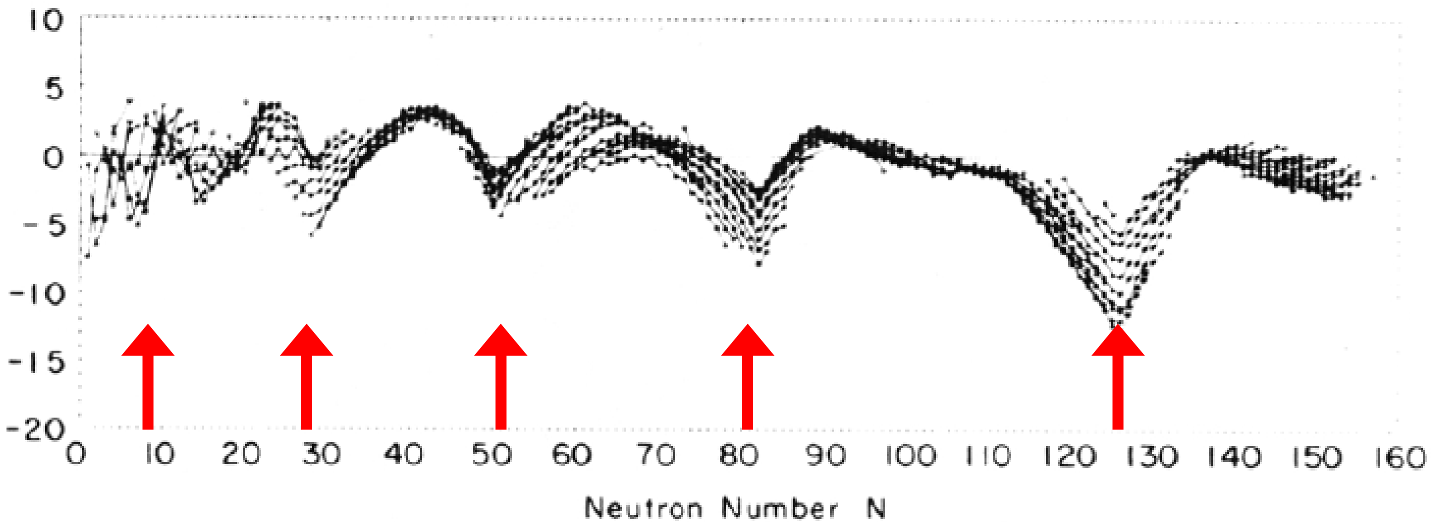
- Shell structure in nuclei and the phenomenological shell model approach to nuclear structure
- Ab-initio approach to nuclear structure. Green's function Monte-Carlo and No-Core Shell-Model.

Roadmap for Theory of Nuclei



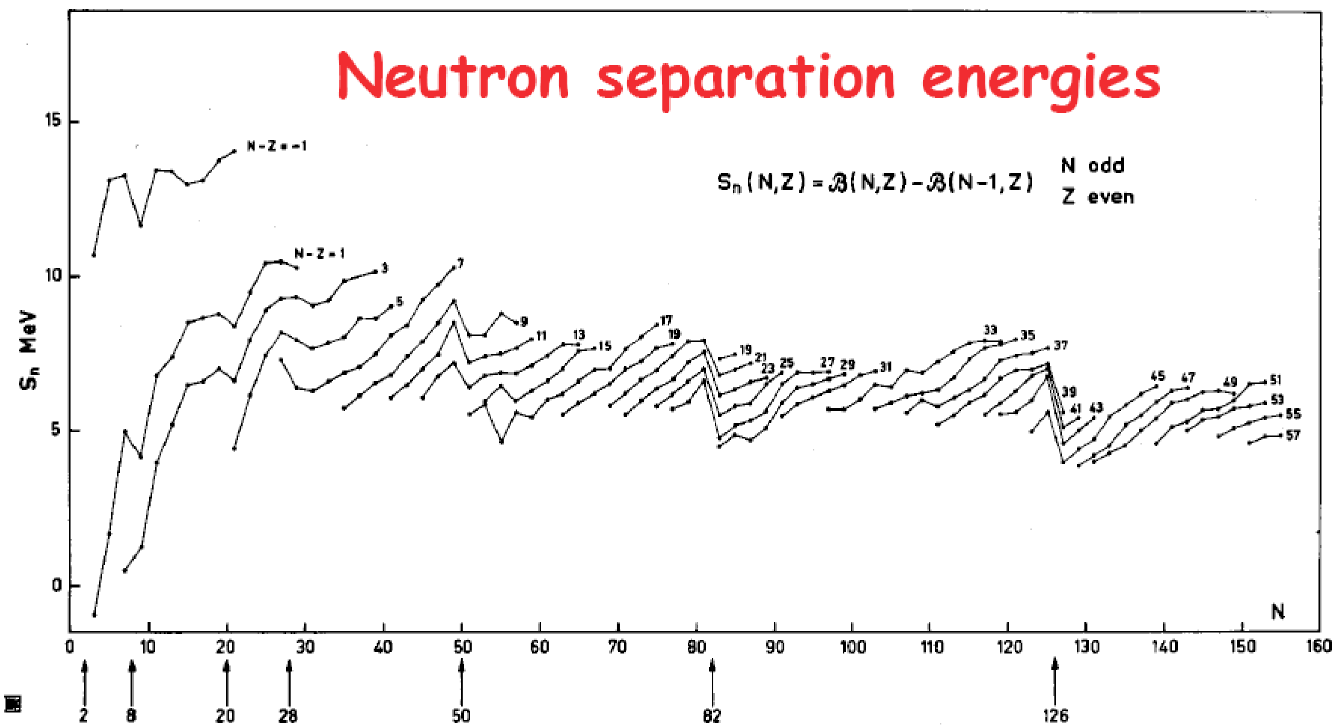
Main goal : To arrive at at comprehensive description of all nuclei and low-energy reactions from the basic interactions between the constituent nucleons

Shell structure in nuclei



From W.D. Meyers and W.J. Swiatecki, Nucl. Phys. 81, 1 (1966).

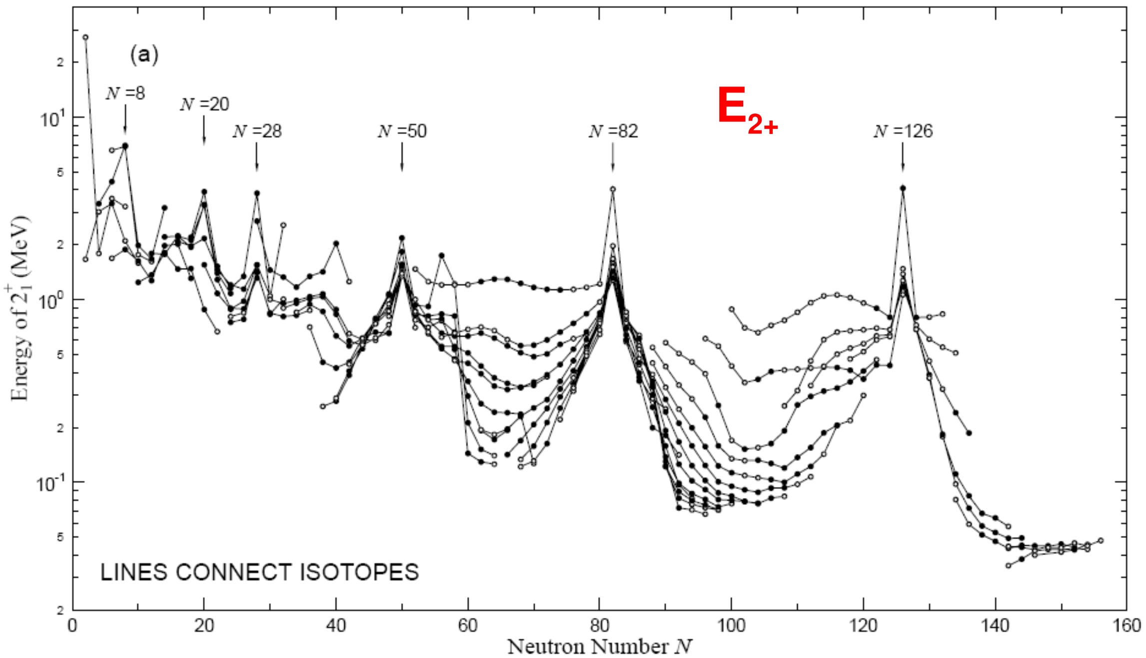
Mass differences: Liquid drop – experiment. Minima at closed shells.

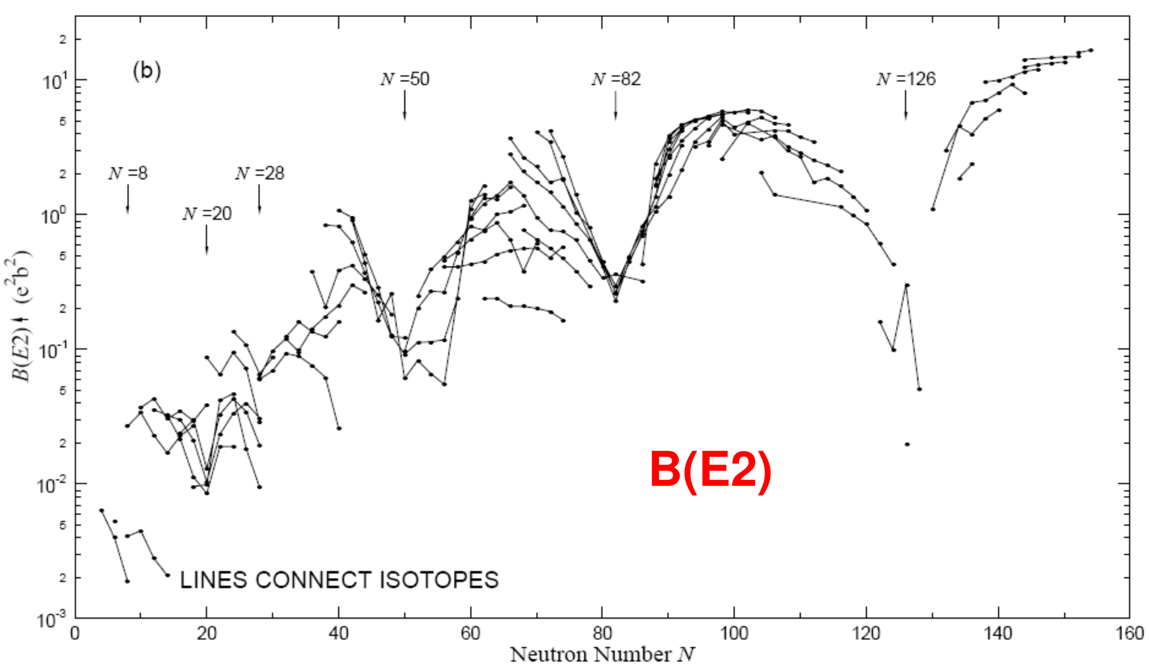


Relatively expensive to remove a neutron form a closed neutron shell.

Bohr & Mottelson, Nuclear Structure.

Shell structure cont'd





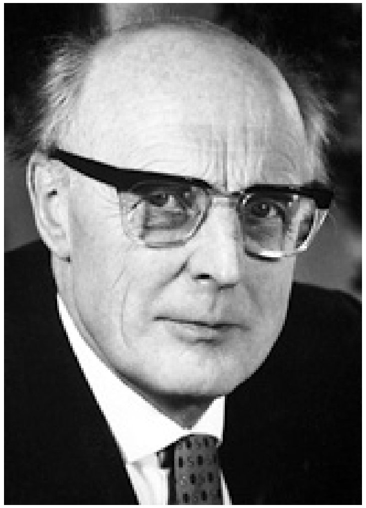
Nuclei with magic N

- Relatively high-lying first 2^+ exited state
- Relatively low B(E2) transition strength

S. Raman et al, Atomic Data and Nuclear Data Tables 78 (2001) 1.

1963 Nobel Prize in Physics

Maria Goeppert-Mayer



J. Hans D. Jensen

“for their discoveries concerning nuclear shell structure”

Magic numbers

$$H_0 = \sum_{i=1}^{A}\left(\frac{\hbar^2}{2M}\nabla^2 + \frac{m}{2}\omega^2 r^2\right)$$

$$E = \hbar\omega(2n + l - 1/2)$$

$$\varphi_{nlm}(\vec{r}) = R_{nl}(r)Y_{lm}(\vartheta,\varphi)$$

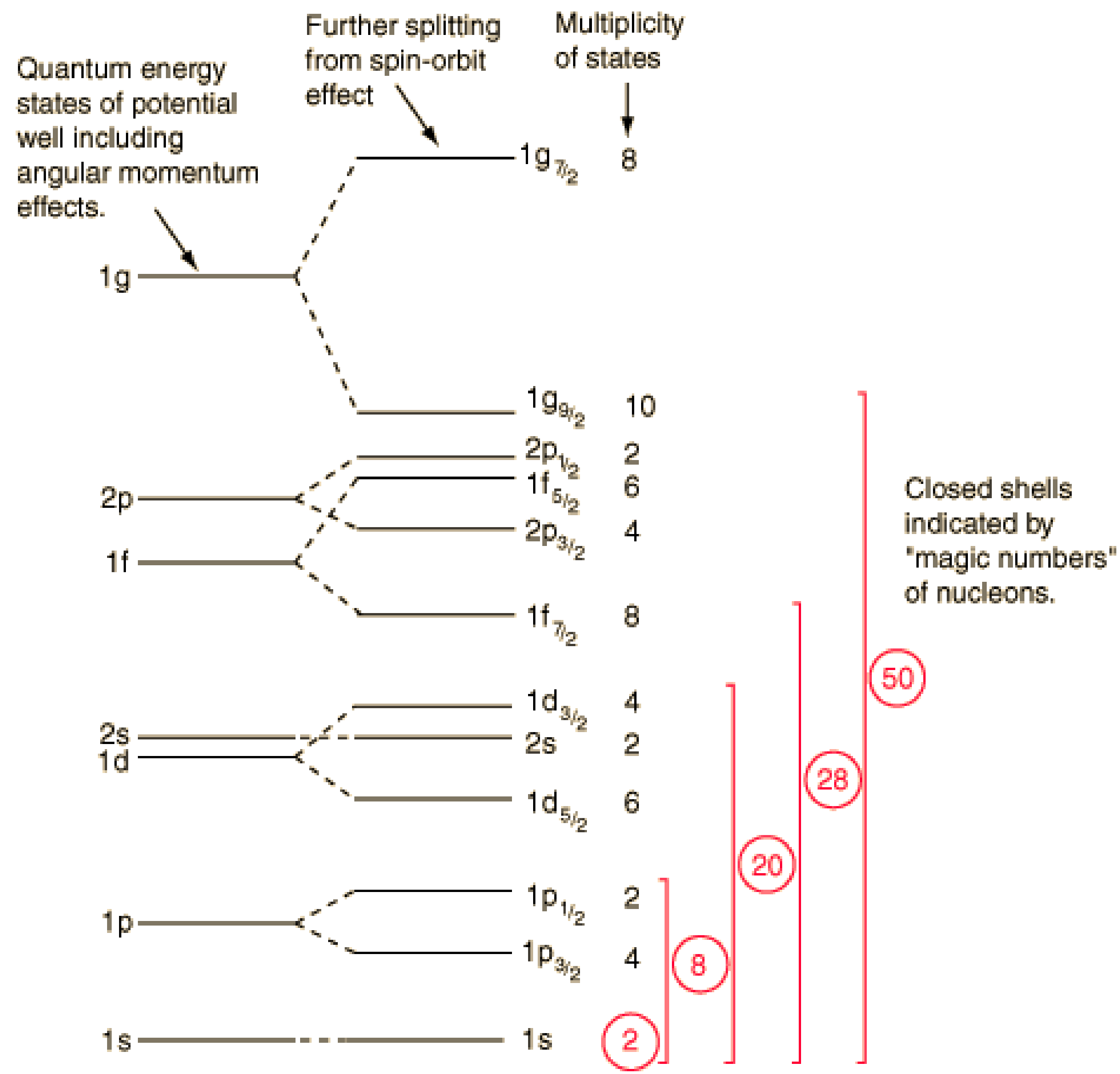
$$R_{nl}(r) \sim r^l e^{-r^2} \times \left[\text{hypergeometric function in } r^2\right]$$

$$\varphi_{nljj_z}(\vec{r}) = R_{nl}(r)\left[Y_l(\hat{r}) \otimes \chi^S_{\frac{1}{2}}\right]^{jj_z}$$

$$\left[Y_l(\hat{r}) \otimes \chi^S_{\frac{1}{2}}\right]^{jj_z} = \sum_{ms_z}\left(lm\tfrac{1}{2}s_z \mid jj_z\right)Y_{lm}(\hat{r})\chi^S_{\frac{1}{2}s_z}$$

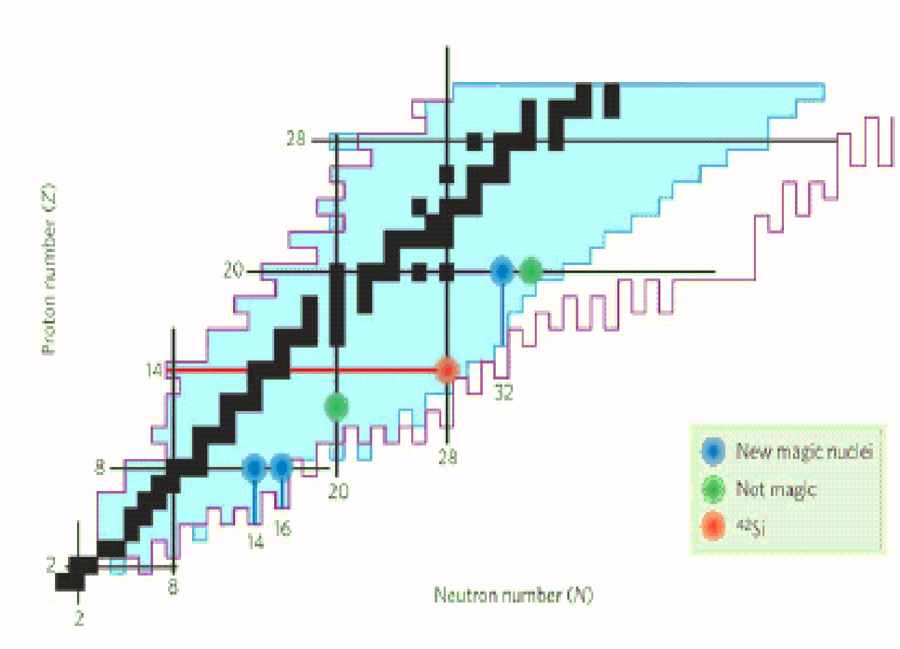
$$\left|\phi_\alpha\right\rangle = \left|nljmt_z\right\rangle \qquad j = l + 1/2$$

Need spin-orbit force to explain magic numbers beyond 20.



$$H_{SM} = \sum_{i=1}^{A}\left(\frac{\hbar^2}{2M}\nabla^2 + \frac{m}{2}\omega^2 r^2 + \eta_l \vec{l}^2 + \xi_{ls}\vec{l} \bullet \vec{s}\right)$$

Does shell structure change in neutron rich nuclei ?



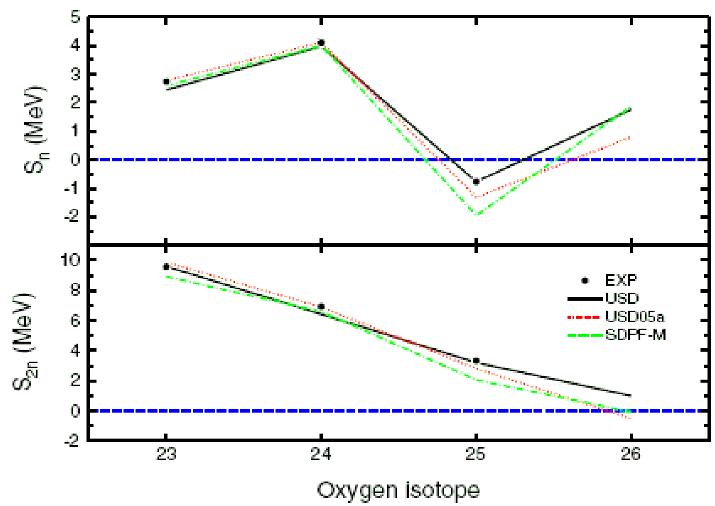


FIG. 4 (color online). The experimental [25,26] (data points) and theoretical [13–15] (lines) one- and two-neutron separation energies for the $N = 15$–18 oxygen isotopes. The experimental error is shown if it is larger than the symbol size.

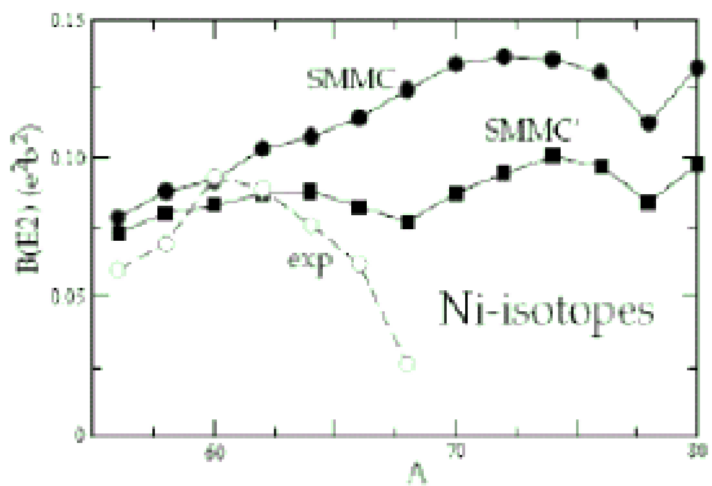
Answer: Yes Indeed! Magic numbers fluctuate when one moves away from stability !!!

C.R. Hoffman PRL, 100, 152502 (2008)
Fridmann et al. Nature 435, 922 (2005)
(comment) Jansens, Nature 435, 207(2005)

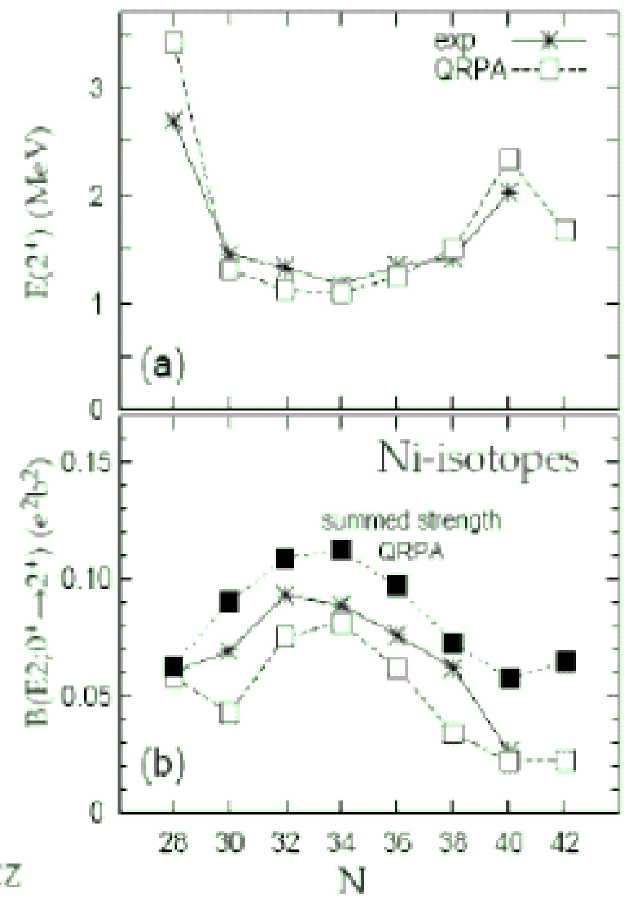
How magic is the magic nucleus 68Ni ?

- -low-lying 0^+_2 level
- -higher energy of the 2^+_1
- -small value of $B(E2, 0^+_1 \to 2^+_1)$

interpreted as evidence for magicity!



Shell-model Monte-Carlo total summed B(E2) strength to the 2^+ excited states.

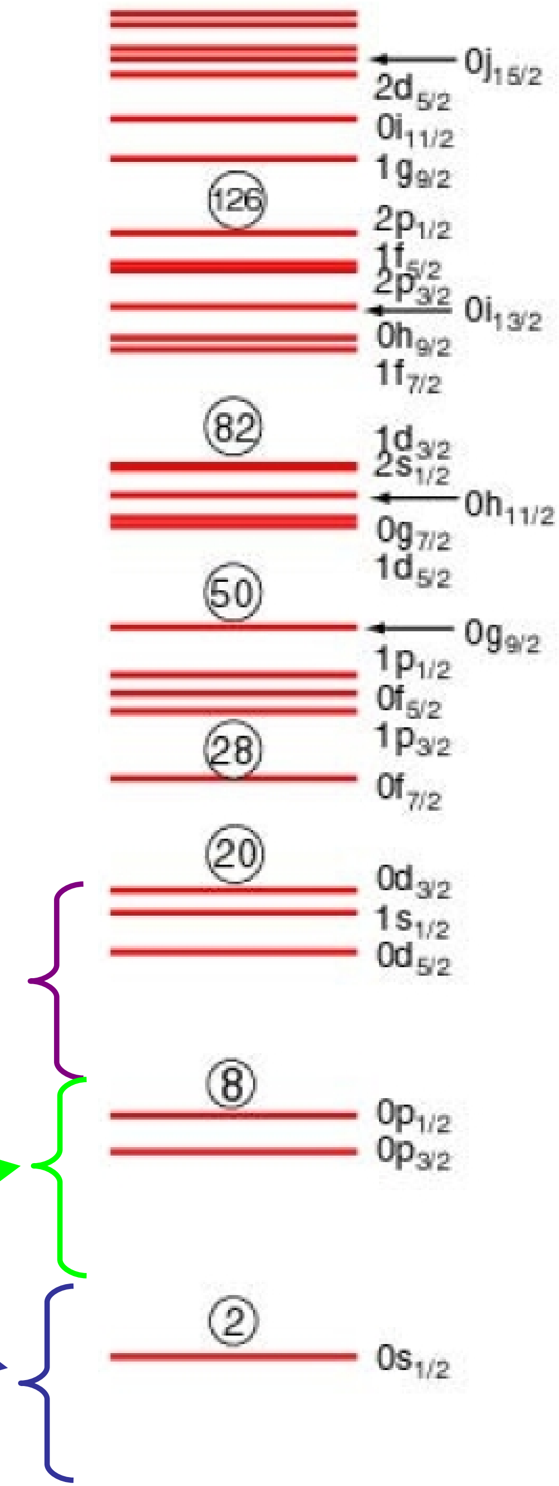


LANGANKE, TERASAKI, NOWACKI, DEAN, AND NAZAREWICZ
PHYSICAL REVIEW C 67, 044314

Traditional shell model

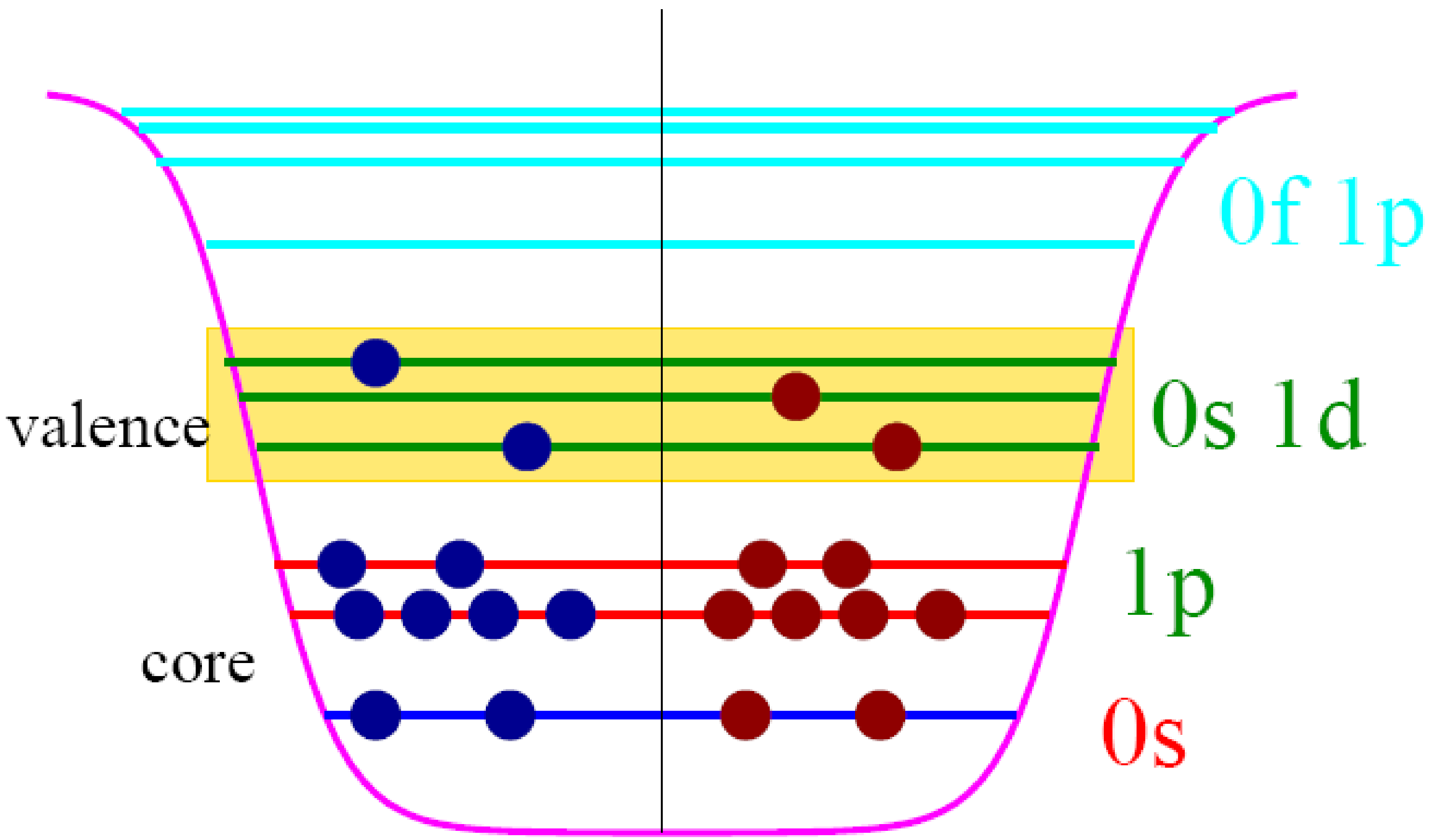
Main idea: Use shell gaps as a truncation of the model space.

- Nucleus (N,Z) = Double magic nucleus (N^*, Z^*)
  + valence nucleons ($N-N^*$, $Z-Z^*$)
- Restrict excitation of valence nuclons to one oscillator shell.
  - Problematic: Intruder states and core excitations not contained in model space.
- Examples:
  - pf-shell nuclei: ^{40}Ca is doubly magic
  - sd-shell nuclei: ^{16}O is doubly magic
  - p-shell nuclei: ^{4}He is doubly magic



Shell model

Example: ^{20}Ne



Shell-model Hamiltonian

Hamiltonian governs dynamics of valence nucleons; consists of one-body part and two-body interaction:

$$\hat{H} = \sum_j \varepsilon_j \hat{a}_j^\dagger \hat{a}_j + \sum_{JTj_1j_2j_1'j_2'} \langle j_1 j_2 | \hat{V} | j_1' j_2' \rangle_{JT} \hat{A}^\dagger_{JT;j_1j_2} \hat{A}_{JT;j_1'j_2'}$$

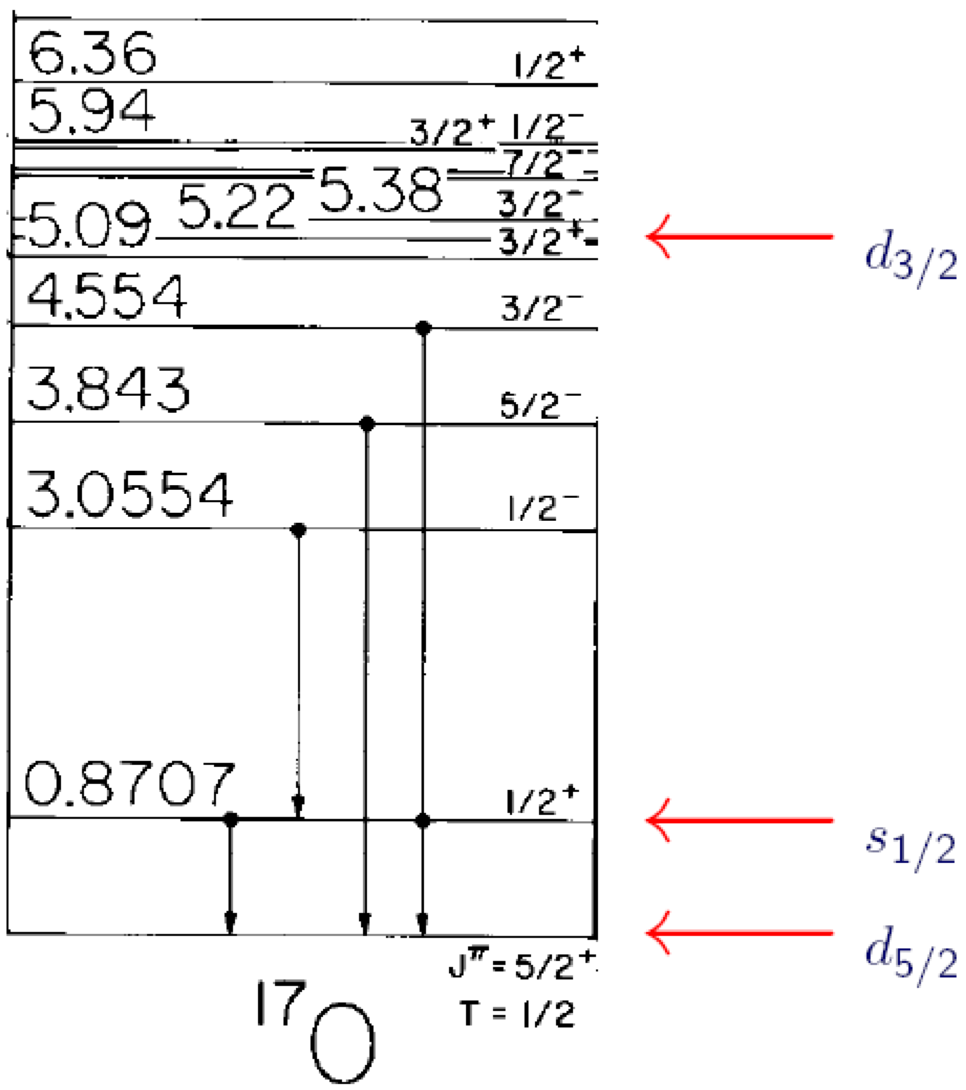
Single-particle energies (SPE)

Two-body matrix elements (TBME) coupled to good spin and isospin

Annihilates pair of fermions

Q: How does one determine the SPE and the TBME?

Empirical determination of SPE and TBME



- Determine SPE from neighbors of closed shell nuclei having mass

  A = closed core +1

- Determine TBME from nuclei with mass

  A = closed core + 2.

- The results of such Hamiltonians become inaccurate for nuclei with a larger number of valence nucleons.
- Thus: More theory needed.

Effective shell-model interaction: G-matrix

- Start from a microscopic high-precision two-body potential
- Include in-medium effects in G-matrix
- Bethe-Goldstone equation

Pauli operator blocks occupied states (core)

$$G = V + V \frac{Q_P}{E - H_0} G$$

microscopic bare interaction

Single-particle Hamiltonian

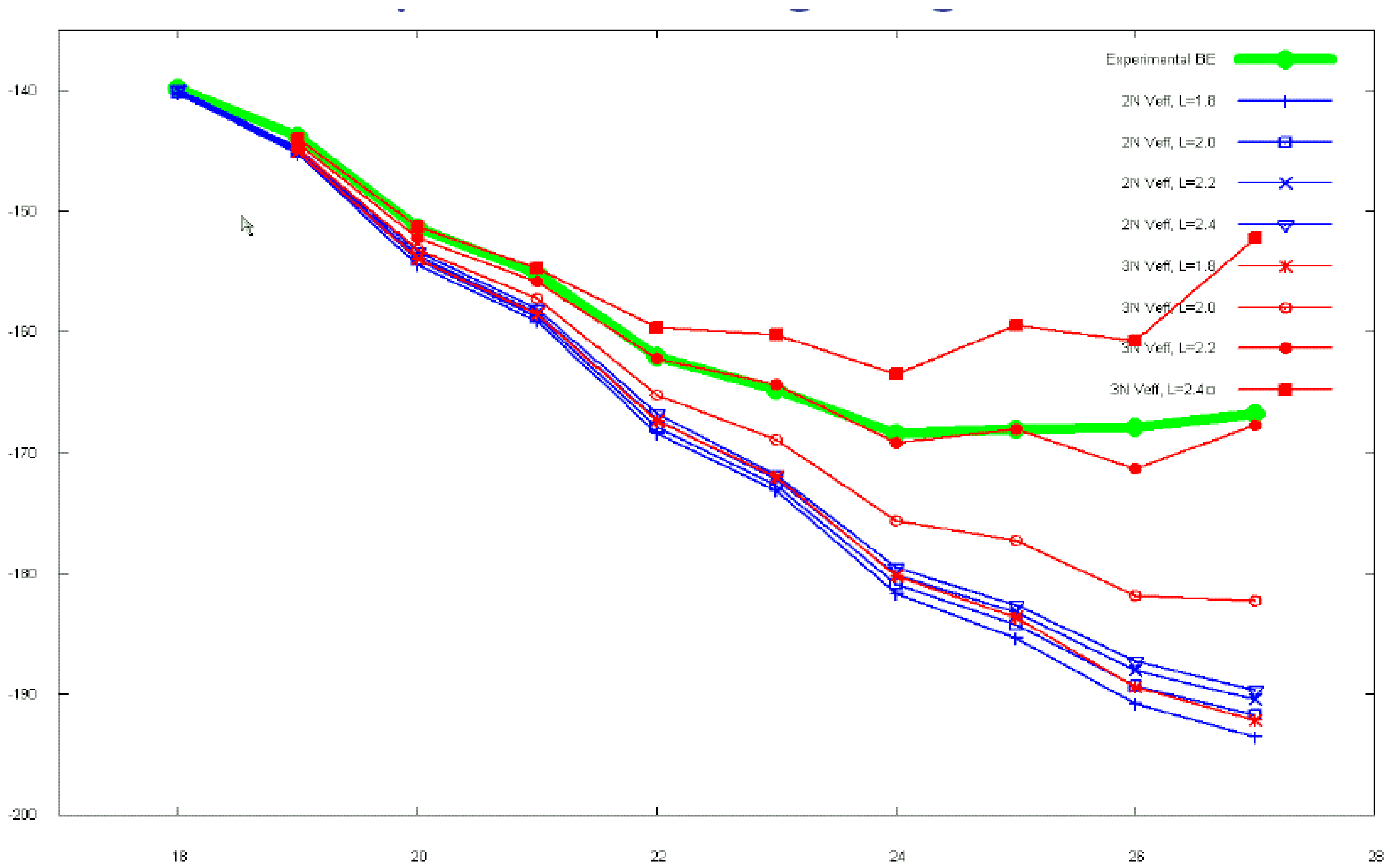
- Formal solution:

$$G = \frac{V}{1 - V Q_P / (E - H_0)}$$

- Properties: in-medium effects renormalize hard core.

See, e.g. M. Hjorth-Jensen et al, Phys. Rep.261 (1995) 125.

Shell model calculations of Oxygen isotopes using v-lowk and effective 2- and 3-body forces.



From Maxim Kartamyshev

Shell-model results for neutron-rich pf-shell nuclei.

Subshell closure at neutron number N=32 in neutron rich pf-shell nuclei (enhanced energy of excited 2^+ state).

No new N=34 subshell.

S. N. Liddick et al, PRL 92 (2004) 072502.

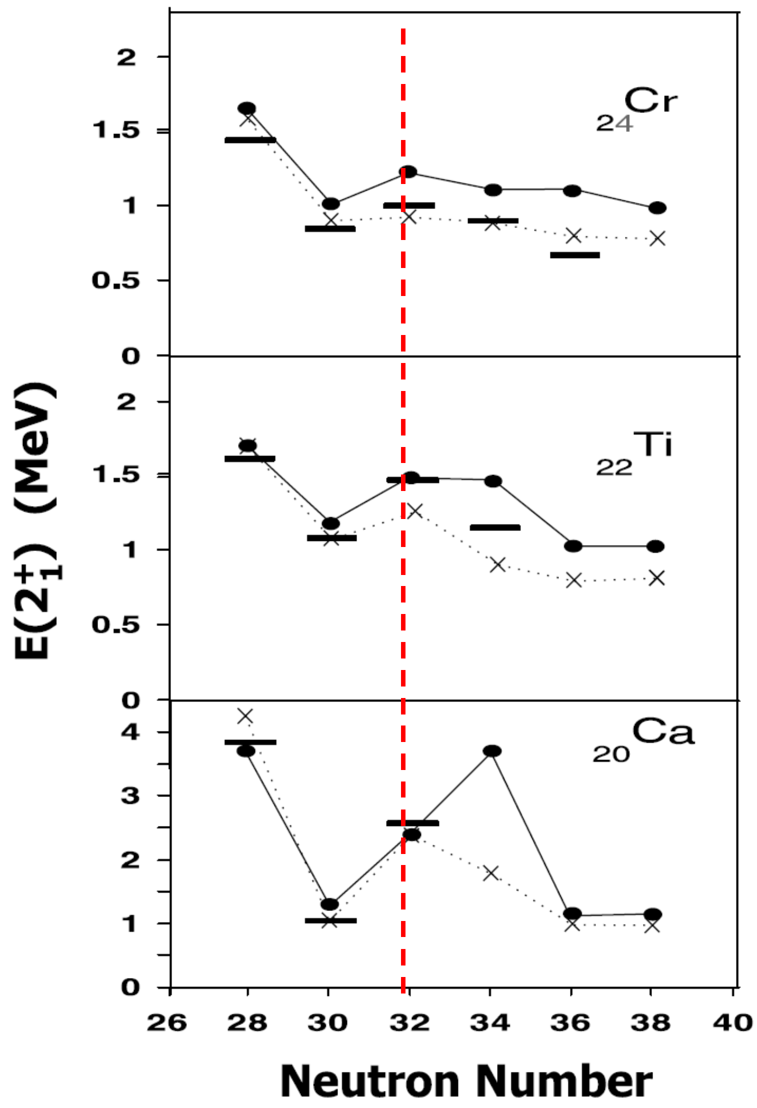


FIG. 3. $E(2_1^+)$ values versus neutron number for the even-even $_{24}$Cr, $_{22}$Ti, and $_{20}$Ca isotopes. Experimental values are denoted by dashes. Shell model calculations using the GXPF1 [14] and KB3G [22] interactions are shown as filled circles and crosses, respectively.

Solving the ab-initio quantum many-body problem

Exact or virtually exact solutions available for:

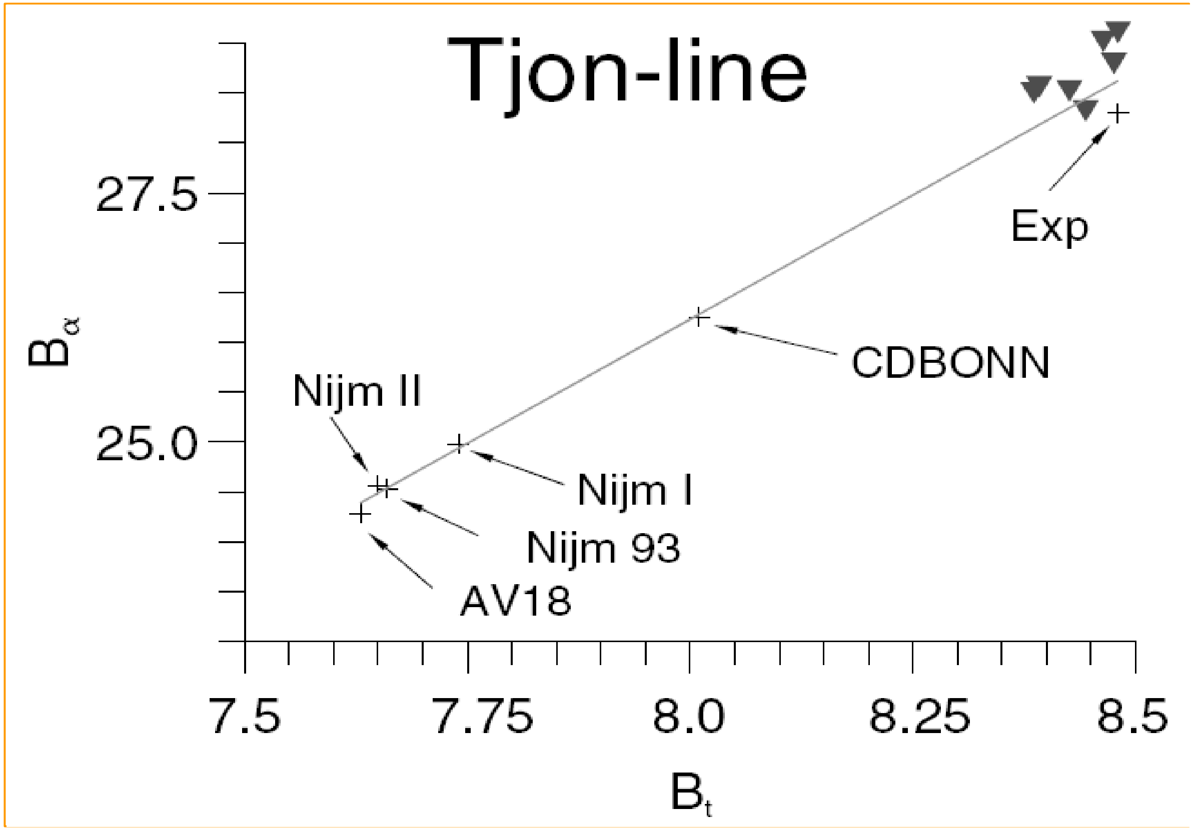
- A=3: solution of Faddeev equation.
- A=4: solvable via Faddeev-Yakubowski approach.
- Light nuclei (up to A=12 at present): Green's function Monte Carlo (GFMC); virtually exact; limited to certain forms of interactions.

Highly accurate approximate solutions available for:

- Light nuclei (up to A=16 at present): No-core Shell model (NCSM); truncation in model space.
- Light and medium mass region (A=4, 16, 40 at present): Coupled cluster theory; truncation in model space and correlations.

1990s: High precision NN potential **models**

- Phenomenological models based on meson exchange.
- Contain about 40 parameters; determined by fit to phase shifts/deuteron.
- Reproduce NN phase shifts with a χ^2/datum very close to 1.0.
- "Nearly perfect" two-body physics.



A. Nogga et al, PRL 85 (2000) 944

Different two-body potential models disagree on structure of triton and alpha particle.

With additional **three-nucleon forces**, agreement with experiment is possible.

(Three-nucleon force differs for different two-body potentials.)

Four-body forces very small.

Green's Function Monte Carlo

Idea:

2. Determine accurate approximate wave function via variation of the energy (The high-dimensional integrals are done via Monte Carlo integration).

$$E = \frac{\langle \Psi_{\text{trial}} | \hat{H} | \Psi_{\text{trial}} \rangle}{\langle \Psi_{\text{trial}} | \Psi_{\text{trial}} \rangle}$$

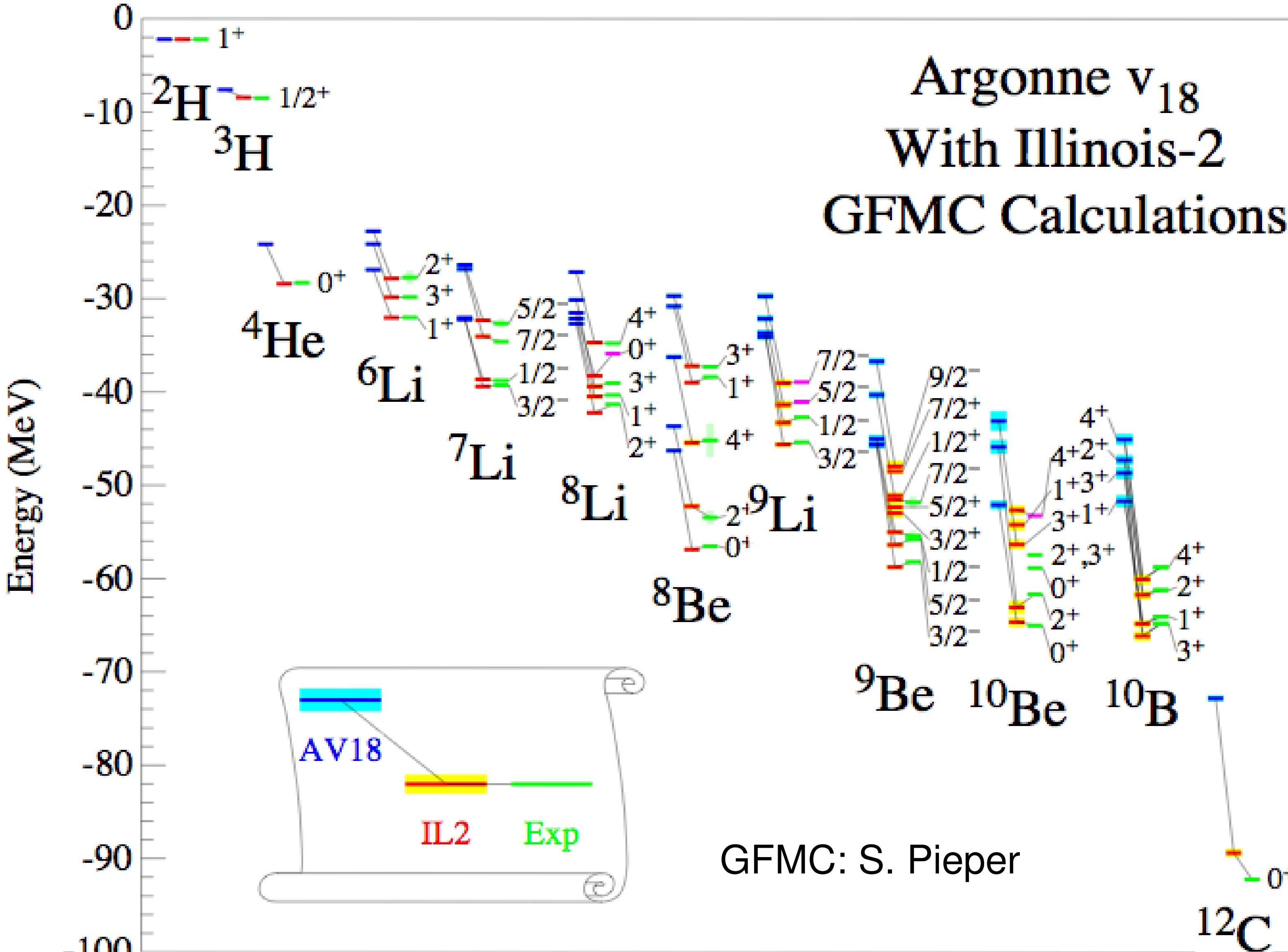
6. Refine wave function and energy via projection with Green's function

$$|\Psi\rangle = \tau \overset{\lim}{\longrightarrow} \infty \; e^{-\tau(\hat{H}-E)} |\Psi_{\text{trial}}\rangle$$

☺ Virtually exact method.

☹ Limited to certain forms of Hamiltonians; computationally expensive method.

GFMC results for light nuclei



1-2% calculations of A = 6 – 12 nuclear energies are possible
excited states with the same quantum numbers computed

GFMC calculations of n-α scattering

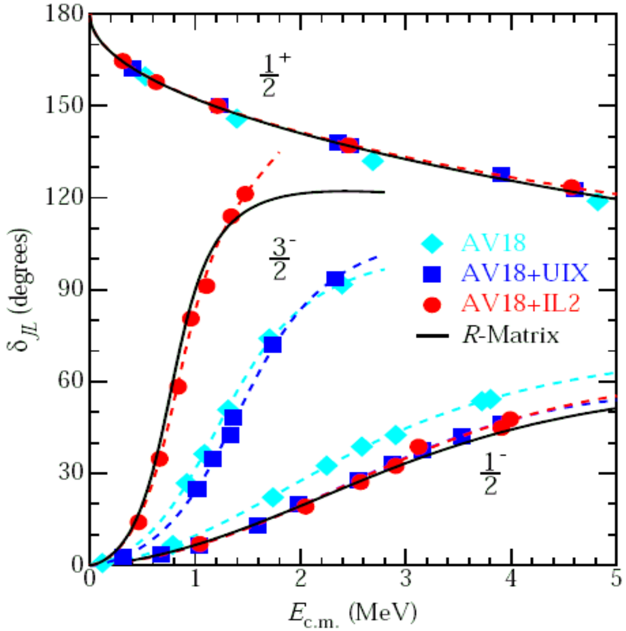


FIG. 1: (Color online) Phase shifts for n-α scattering. Filled symbols (with statistical errors smaller than the symbols) are GFMC results; dashed curves are fits described in the text; and solid curves are from an R-matrix fit to data [14].

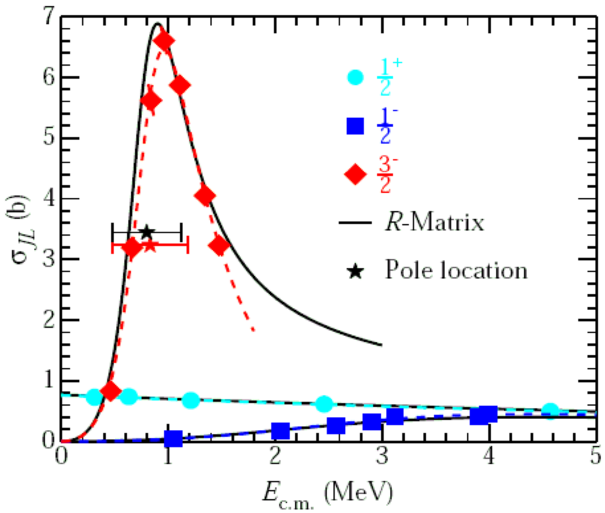


FIG. 2: (Color online) Calculated and R-matrix partial-wave cross sections. The calculations, shown with their Monte Carlo error bars, are for the AV18+IL2 Hamiltonian. Stars show the pole energies in $3/2^-$ scattering for the R-matrix fit and for AV18+IL2, with the bars indicating the imaginary part.

p3/2 resonance : 0.83 - 0.35i (exp 0.798 - 0.324i)
p1/2 resonance : 2.07 - 2.6i (exp 2.07 - 2.79i)

K. Nellott et al, Phys. Rev. Lett 99, 022502 (2007)

No core shell model

Idea: Solve the A-body problem in a harmonic oscillator basis.

2. Take K single particle orbitals
3. Construct a basis of Slater determinants
4. Express Hamiltonian in this basis
5. Find low-lying states via diagonalization

☺ Get eigenstates and energies

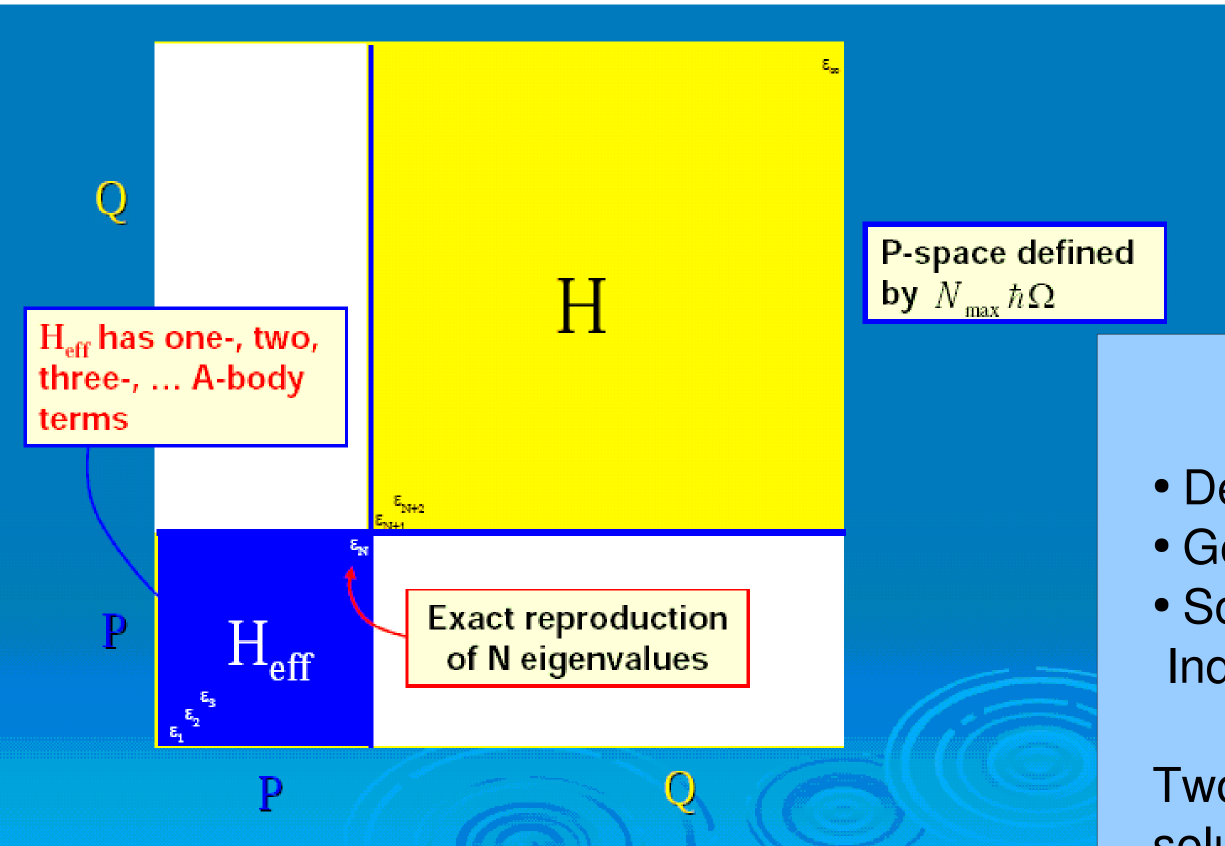
☺ No restrictions regarding Hamiltonian

☹ Number of configurations and resulting matrix very large: There are

$$\binom{K}{A} = \frac{K!}{(K-A)!A!}$$

ways to distribute A nucleons over K single-particle orbitals.

The general idea behind effective interactions and the Lee-Suzuki similarity transformation.



- Define a model space P
- Generate an effective interaction in P
- Solve the many-body problem

Induces many-body forces

Two ways of converging to the bare solution :

1. Increase model space P until convergence
2. Include induced many-body forces, reproduces bare solution

Working in a finite model space

NCSM and Coupled-cluster theory solve the Schroedinger equation in a model space with a *finite* (albeit large) number of configurations or basis states.

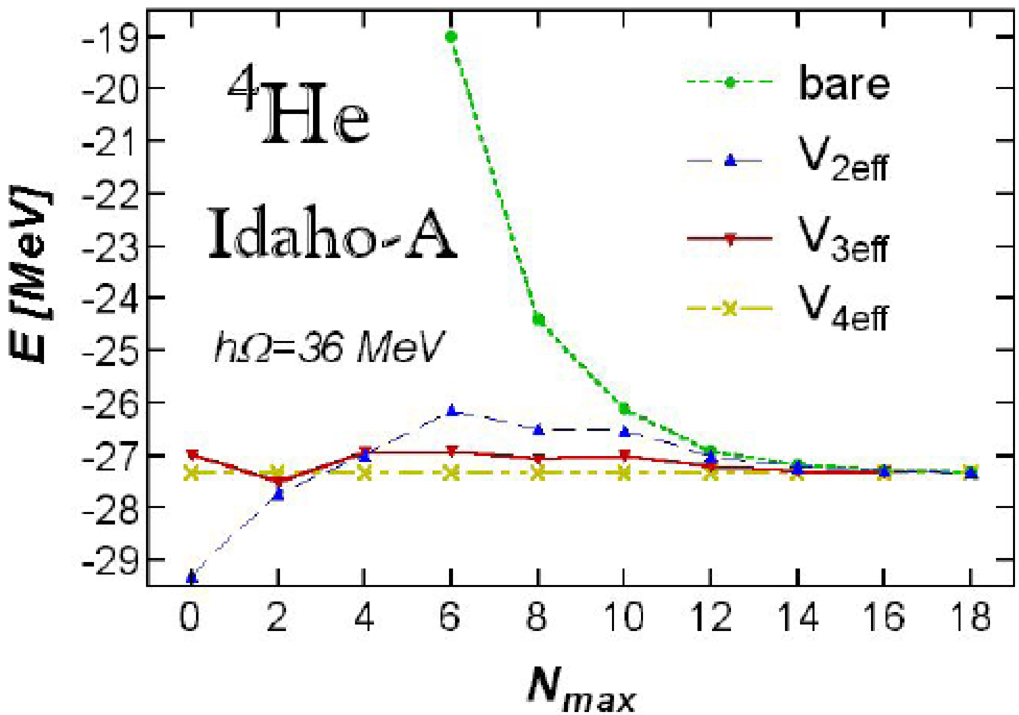
Problem: High-momentum components of high-precision NN interactions require enormously large spaces.

Solution: Get rid of the high-momentum modes via a renormalization procedure. (Vlow-k is an example)

Price tag:

Generation of 3, 4, …, A-body forces unavoidable.

Observables other than the energy also need to be transformed.



E. Ormand
http://www.phy.ornl.gov/npss03/ormand2.ppt

Theorists agree with each other

PHYSICAL REVIEW C, VOLUME 64, 044001

Benchmark test calculation of a four-nucleon bound state

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(Received 20 April 2001; published 27 August 2001)

In the past, several efficient methods have been developed to solve the Schrödinger equation for four-nucleon bound states accurately. These are the Faddeev-Yakubovsky, the coupled-rearrangement-channel Gaussian-basis variational, the stochastic variational, the hyperspherical variational, the Green's function Monte Carlo, the no-core shell model, and the effective interaction hyperspherical harmonic methods. In this
NN interaction. The results of all schemes agree very well showing the high accuracy of our present ability to calculate the four-nucleon bound state.

TABLE I. The expectation values $\langle T \rangle$ and $\langle V \rangle$ of kinetic and potential energies, the binding energies E_b in MeV, and the radius in fm.

Method	$\langle T \rangle$	$\langle V \rangle$	E_b	$\sqrt{\langle r^2 \rangle}$
FY	102.39(5)	−128.33(10)	−25.94(5)	1.485(3)
CRCGV	102.30	−128.20	−25.90	1.482
SVM	102.35	−128.27	−25.92	1.486
HH	102.44	−128.34	−25.90(1)	1.483
GFMC	102.3(1.0)	−128.25(1.0)	−25.93(2)	1.490(5)
NCSM	103.35	−129.45	−25.80(20)	1.485
EIHH	100.8(9)	−126.7(9)	−25.944(10)	1.486

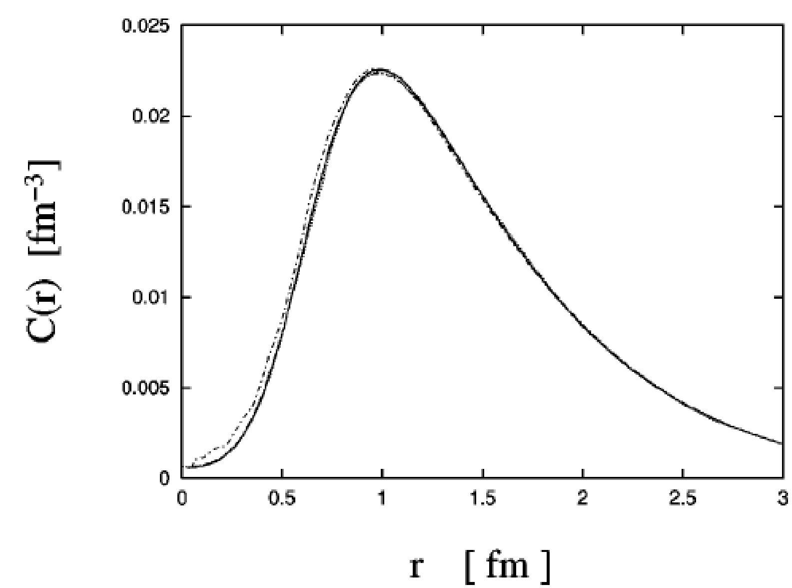
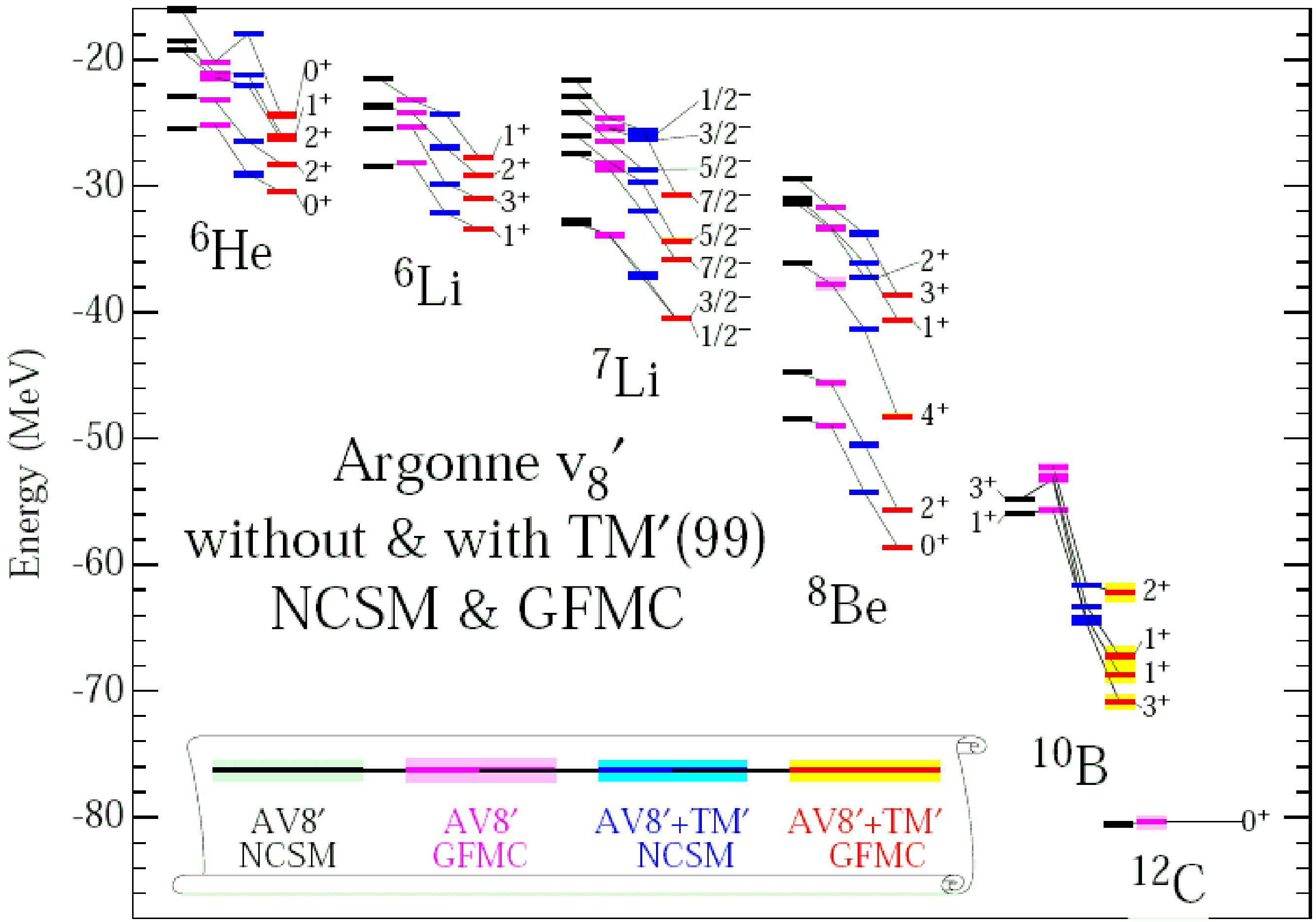


FIG. 1. Correlation functions in the different calculational schemes: EIHH (dashed-dotted curves), FY, CRCGV, SVM, HH, and NCSM (overlapping curves).

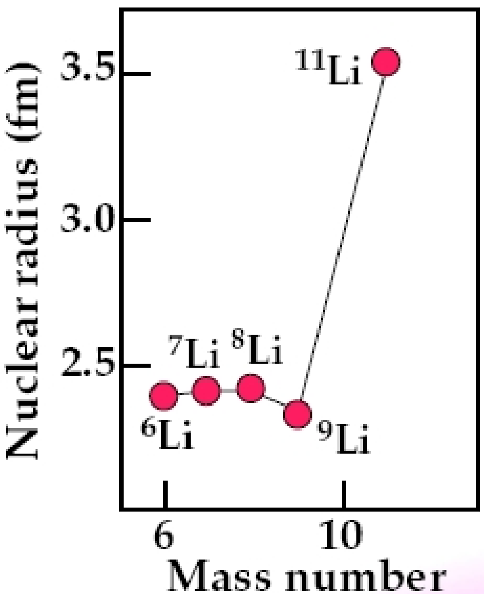
Comparison between NCSM and GFMC



S. Pieper Nucl. Phys. A 751 (2005) 516-532

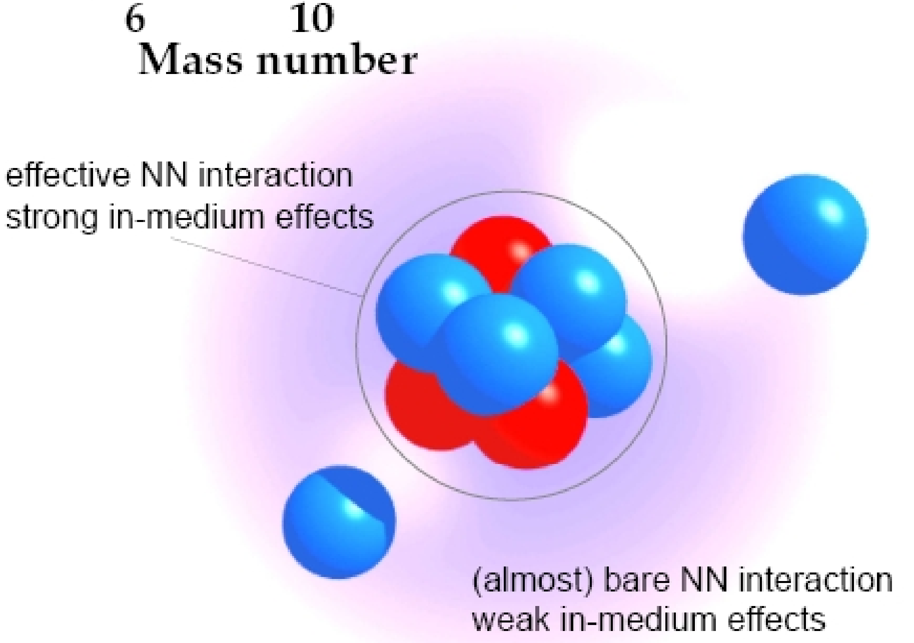
Figure 6. Comparison of NCSM and GFMC energies for the AV8′ and AV8′+TM′ Hamiltonians.

Ab-initio calculations of charge radii of Li isotopes



I. Tanihata et al.
Phys. Rev. Lett. 55, 2676 (1985)

Interaction cross section
measurements at Bevalac
(790 MeV/u)



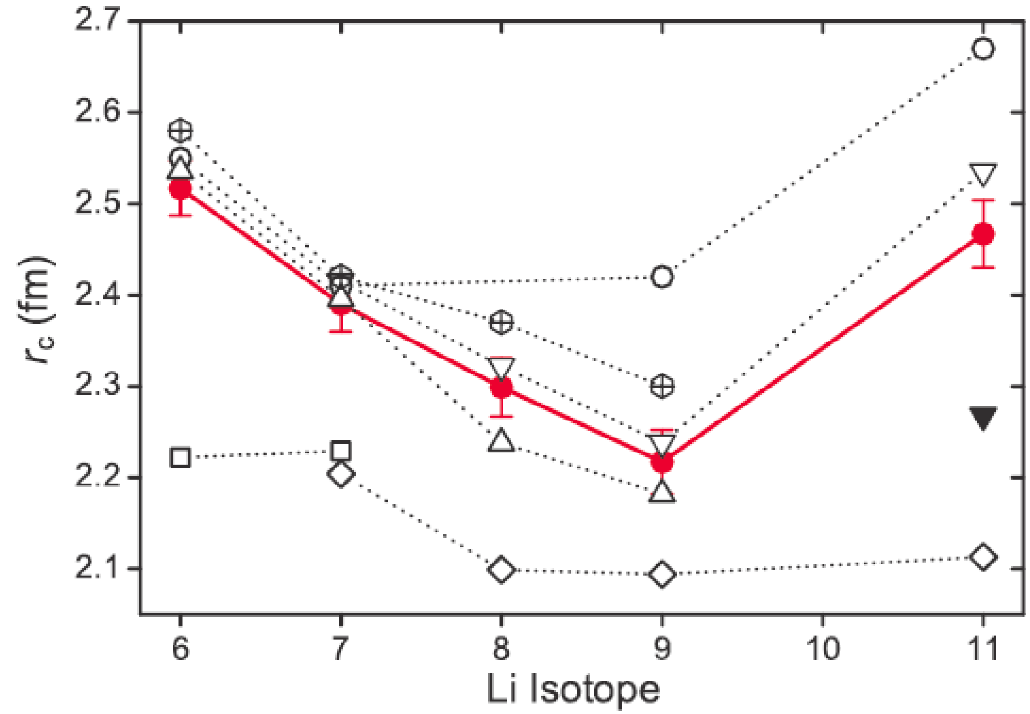
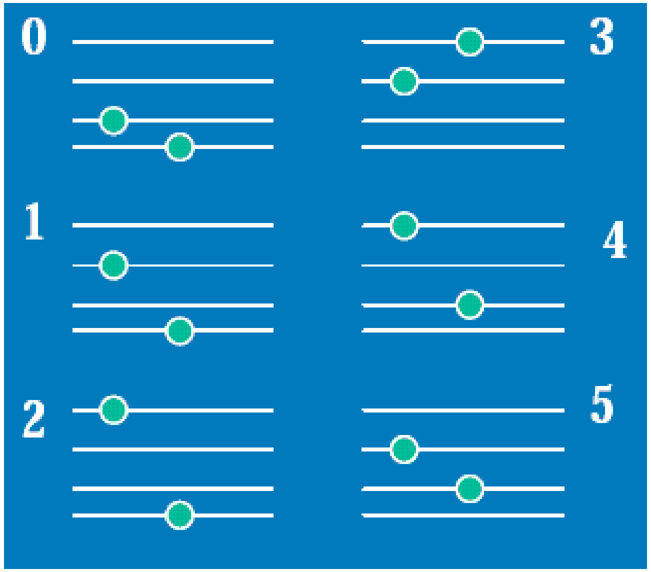


FIG. 2 (color online). Experimental charge radii of lithium isotopes (red, ●) compared with theoretical predictions: △: GFMC calculations [4,22], ▽: SVMC model [27,28] (▼: assuming a frozen ^{9}Li core), ⊕: FMD [26], ○: DCM [19], □ and ◇: *ab initio* NCSM [23,24].

R. Sanchez et al, PRL 96 (2006) 33002.

The N! catastrophe.
Specific example: 2 particles in 4 states



$$I=0 \quad a_2^+ a_1^+ |--\rangle = |1100\rangle = |\Phi_0\rangle$$
$$I=1 \quad a_3^+ a_1^+ |--\rangle = |1010\rangle = |\Phi_1\rangle$$
$$I=2 \quad a_4^+ a_1^+ |--\rangle = |1001\rangle = |\Phi_2\rangle$$
$$I=3 \quad a_3^+ a_2^+ |--\rangle = |0110\rangle = |\Phi_3\rangle$$
$$I=4 \quad a_4^+ a_2^+ |--\rangle = |0101\rangle = |\Phi_4\rangle$$
$$I=5 \quad a_4^+ a_3^+ |--\rangle = |0011\rangle = |\Phi_5\rangle$$

Scaling: Number of basis states

Ooops.. These are huge numbers

Problem : How to deal with such large dimensions

$n =$ number of particles;

$N =$ number of single-particle states

$$C(N,n) = \frac{N!}{(N-n)!\,n!}$$

$$C(10,100) = 1.7x10^{13}$$

$$C(1000,100) = 6x10^{139}$$

Summary

- Shell model a powerful tool for understanding of nuclear structure.
- Shell model calculations based on microscopic interactions
  - Adjustments are needed
  - Due to neglected three body forces (?!)
- Effective interactions have reached maturity to make predictions, and to help understanding experimental data
- Green's function Monte-Carlo and No-core Shell-model capable of ab-initio description of nuclei with $A < 12$
- Due to factorial scaling of the method, very difficult to extend to heavier systems.
- Need accurate method with softer scaling in order to extend the ab-initio program to heavier systems.