Non-Relativistic EFTs of QCD

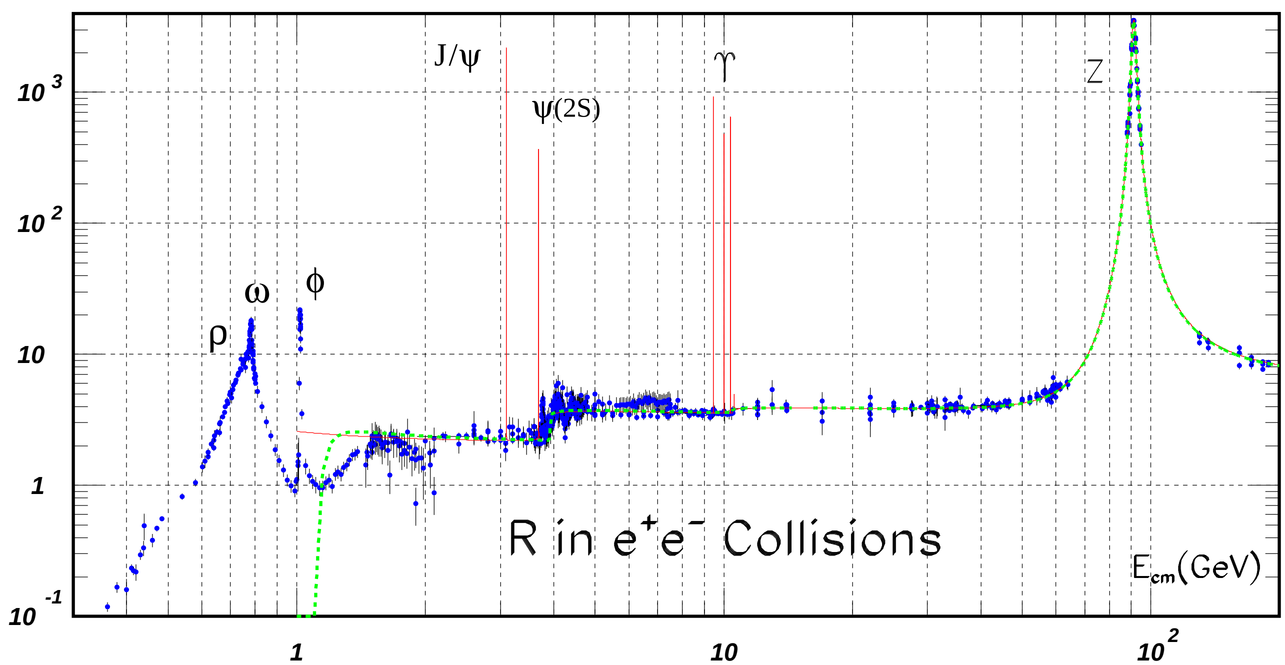
– strong coupling –

Antonio Vairo

INFN and University of Milano

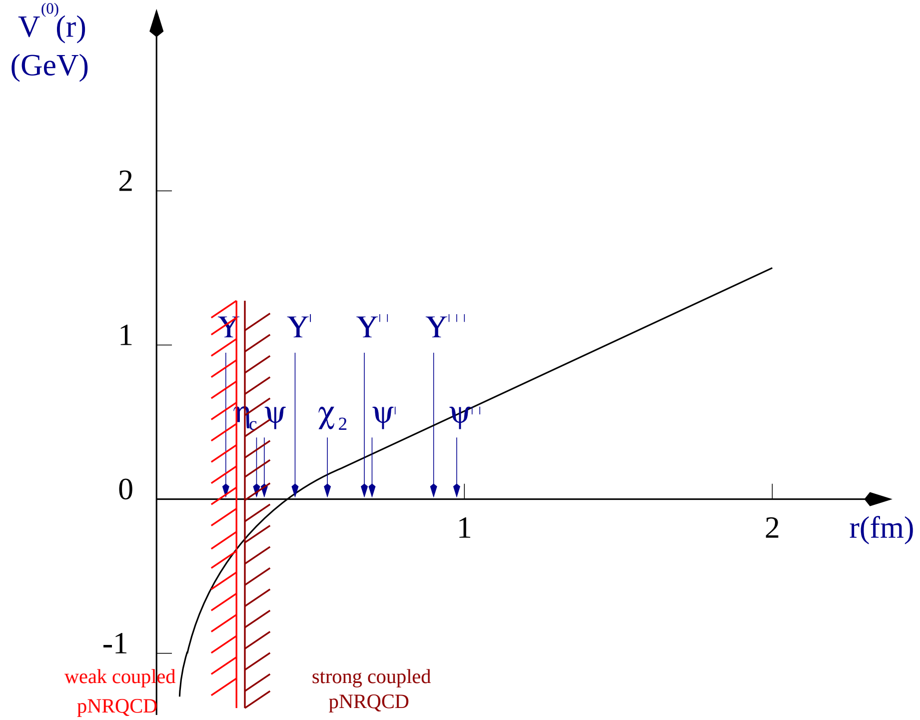
Quarkonium Working Group

Motivations



Motivations

- Most of the heavy quarkonium resonances are in the non-perturbative regime.



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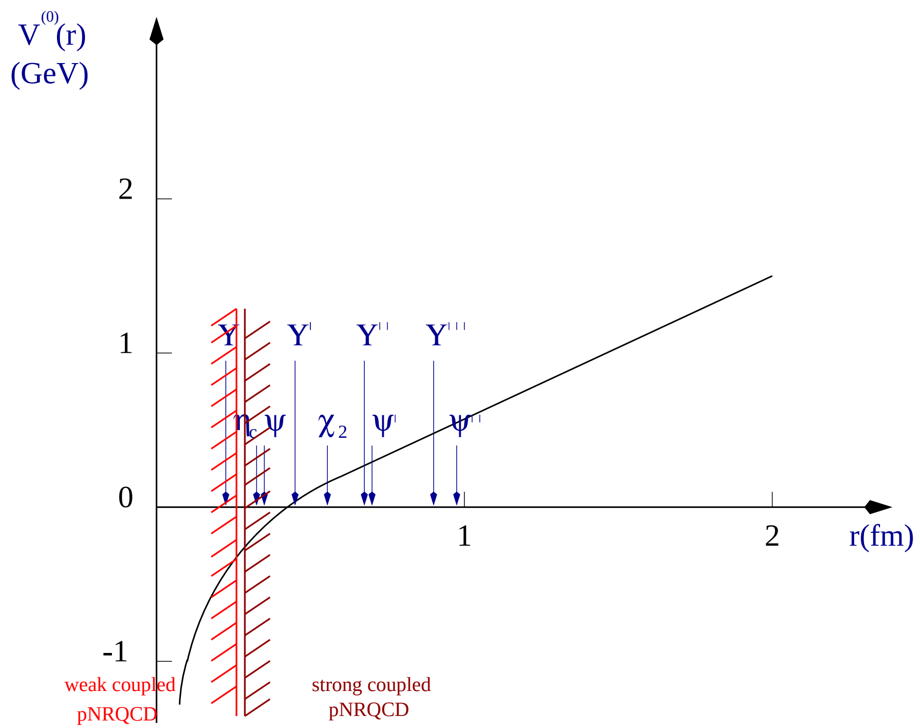
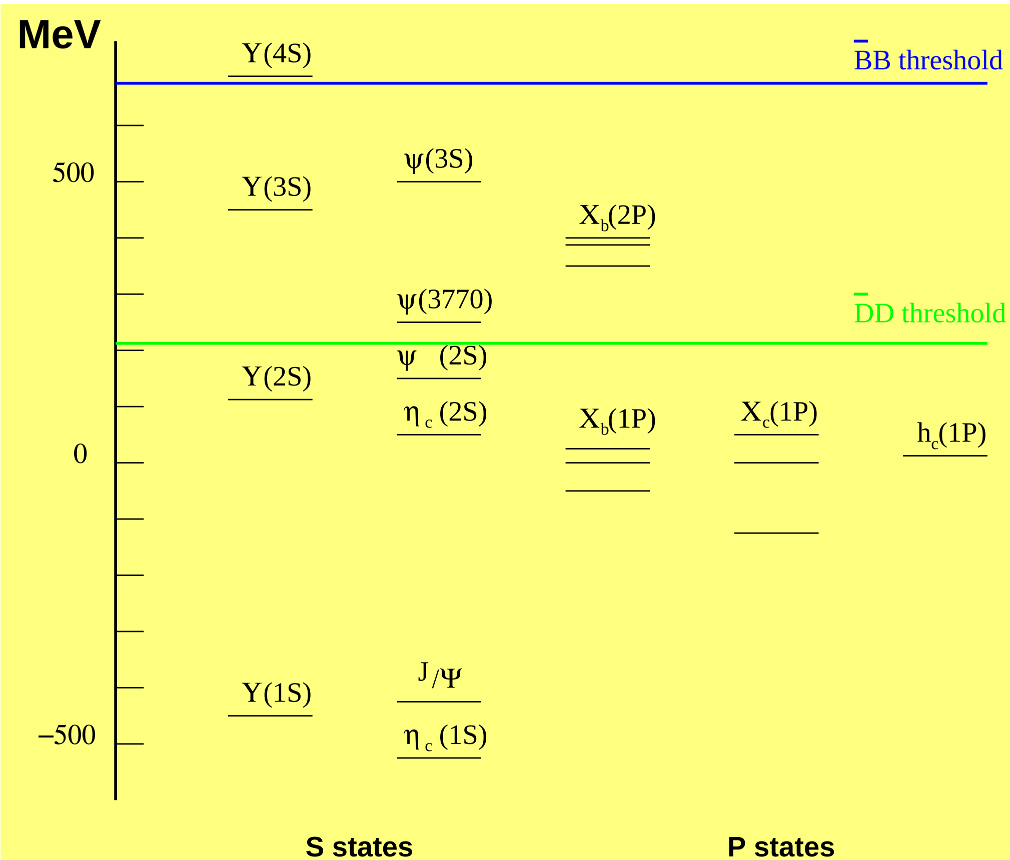


- Quantum Mechanics of a non-Abelian Field Theory .

Quarkonium Scales



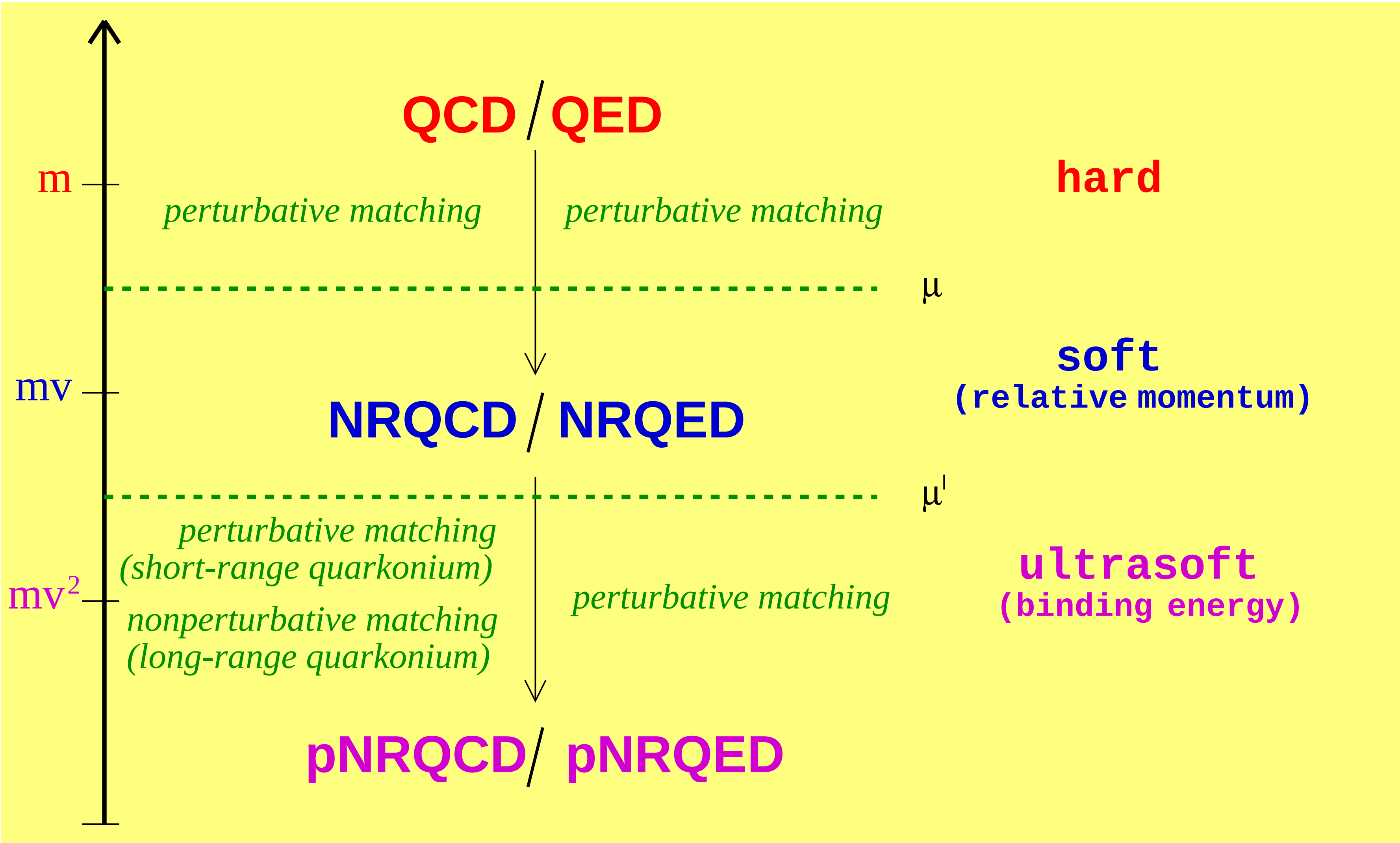
Normalized with respect to $\chi_b(1P)$ and $\chi_c(1P)$

The mass scale is perturbative:
$m_b \simeq 5$ GeV, $m_c \simeq 1.5$ GeV

The system is non-relativistic:
$\Delta_n E \sim mv^2$, $\Delta_{\rm fs} E \sim mv^4$
$v_b^2 \simeq 0.1$, $v_c^2 \simeq 0.3$

The dynamical scales are:
$r \sim 1/mv$, $E \sim mv^2 \qquad v \ll 1$

Non-Relativistic EFT



In QCD another scale is relevant: Λ_{QCD}

NRQCD

NRQCD

$$\mathcal{L} = \psi^\dagger \left(iD_0 + \frac{\mathbf{D}^2}{2m} + c_F \frac{\mathbf{S} \cdot g\mathbf{B}}{m} + c_D \frac{[\mathbf{D}\cdot, g\mathbf{E}]}{8m^2} + \ldots \right) \psi$$

$$+ \chi^\dagger \left(\qquad \ldots \qquad \right) \chi$$

$$+ \sum_K \frac{f}{m^2} \psi^\dagger \, K \, \chi \chi^\dagger \, K \, \psi + \cdots$$

$$- \frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \ldots$$

Caswell Lepage 86, Bodwin Braaten Lepage 95, Manohar 97

(A) NRQCD power counting

* Counting in $\alpha_{\mathrm{s}}(m)$:

$$c_F = 1 - \frac{C_A}{2}\frac{\alpha_{\mathrm{s}}}{\pi}\log\frac{m}{\mu} + \dots \qquad c_S = 2c_F - 1$$

$$c_D = 1 + \left(\frac{2}{3}C_A + \frac{8}{3}C_F\right)\frac{\alpha_{\mathrm{s}}}{\pi}\log\frac{m}{\mu} + \dots$$

$$f = \mathcal{O}(\alpha_{\mathrm{s}}) \qquad \mathrm{Im} f = \mathcal{O}(\alpha_{\mathrm{s}}^2)$$

(A) NRQCD power counting

* Counting in v:

1) $\int d^3\mathbf{x}\,\psi^\dagger\psi \simeq 1 \Rightarrow |\psi|^2 \sim \frac{1}{(\Delta x)^3} \sim m^3 v^3$

2) $K^{(d)} \sim (mv)^d$ (e.g. $g\mathbf{E}, g\mathbf{B} \sim m^2 v^2$, $\mathbf{D} \sim mv$)

3) $D_0 \sim mv^2$ (virial theorem)

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E.g.

S-wave decay up to $\mathcal{O}(mv^5)$: $\Gamma \sim \frac{|\psi(0)|^2}{m^2}\left(1 + \frac{1}{m^2} \times \mathcal{O}(m^2 v^2)\right)$

P-wave decay up to $\mathcal{O}(mv^5)$: $\Gamma \sim \frac{|\psi'(0)|^2}{m^4}$

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3) $D_0 \sim mv^2$ (virial theorem)

- *The power counting is not unique.*

  E.g. in Lepage et al. 92 ("standard NRQCD power counting"):

  $gA_0 \sim mv^2$, $g\mathbf{A} \sim mv^3$, $g\mathbf{E} \sim m^2 v^3$, $g\mathbf{B} \sim m^2 v^4$.

(A) NRQCD power counting

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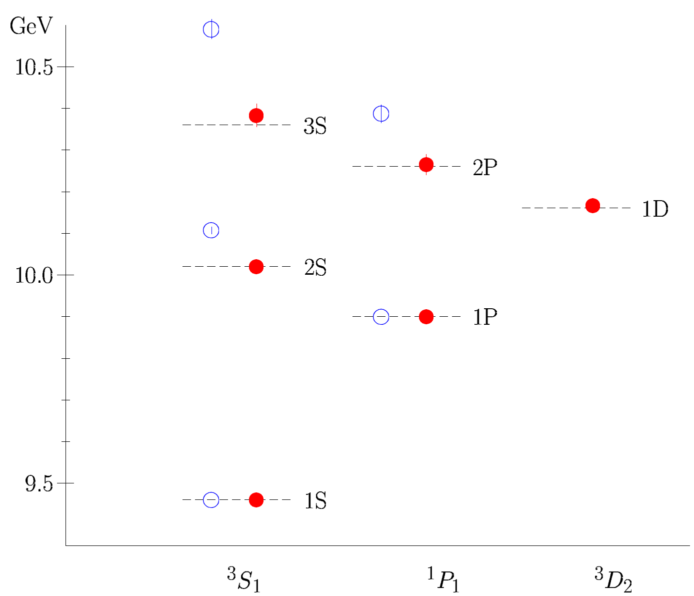
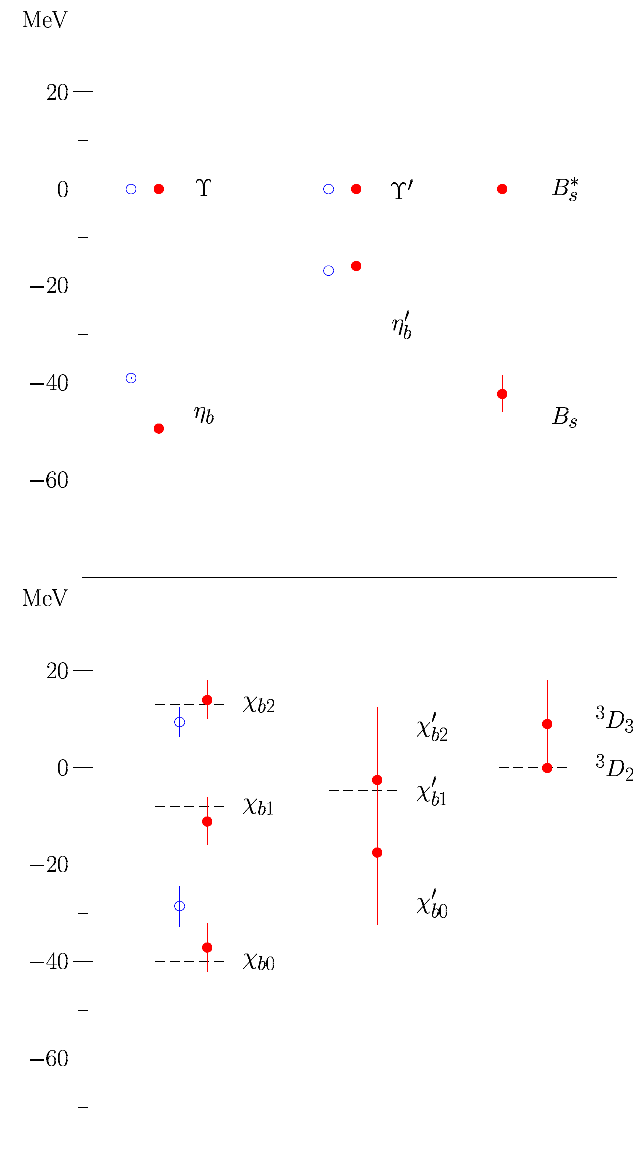
3) $D_0 \sim mv^2$ (virial theorem)

□ *The power counting uses arguments rigorously suited only for bound states in a non-relativistic quantum-mechanical context.*

Lattice NRQCD



Bottomium



<u>Lattice matching coefficients are known at tree level</u>

* Radial splittings up to $\mathcal{O}(\alpha_s v^2) \simeq 0.2 \times 0.1 \simeq 2\%$

* Fine and hf splittings up to $\mathcal{O}(\alpha_s) \simeq 0.2 \simeq 20\%$

HPQCD and UKQCD coll. 02, Davies 02

Decay in the Singlet Model

$$|H\rangle = \int \frac{d^3k}{(2\pi)^3} \Phi^{ij}_{uv}(\mathbf{k}) |Q(\mathbf{k})^{iu} Q(-\mathbf{k})^{jv}\rangle \qquad \Phi^{ij}_{uv}(\mathbf{x}) \sim (\cdots) R(x)$$

$$\Gamma(\chi_0 \to \mathrm{LH}) = \sum_X \Gamma(\chi_0 \to X) \simeq \Gamma(\chi_0 \to gg)$$

Decay in the Singlet Model

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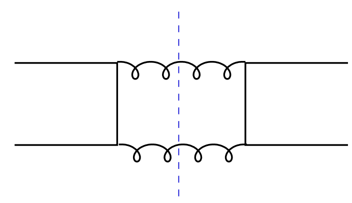
$$\Gamma(\chi_0 \to \mathrm{LH}) \simeq \langle H | 2\ \mathrm{Im}$$



$$|H\rangle$$

Decay in the Singlet Model

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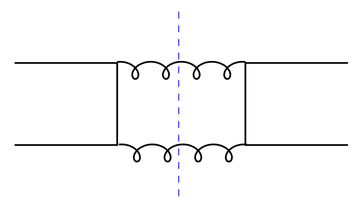
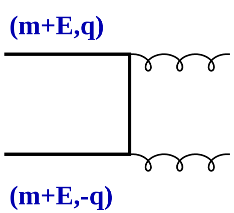
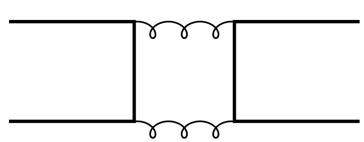
$$\Gamma(\chi_0 \to \mathrm{LH}) \simeq \langle H | 2\ \mathrm{Im}$$  $$|H\rangle$$

- tree level

$$\Bigg|$$ 

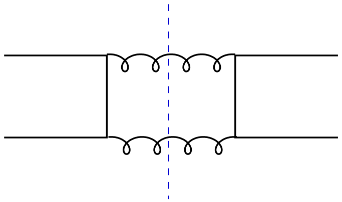
$$\Bigg|^2 = 2\ \mathrm{Im} \Bigg($$  $$\Bigg) \xrightarrow[q \sim mv \ll m]{} \frac{4\pi\alpha_s^2}{3} \frac{\mathbf{S}\cdot\boldsymbol{\nabla}\delta(\mathbf{r})\mathbf{S}\cdot\boldsymbol{\nabla}}{m^4}$$

$$\Gamma = \langle H | \cdots | H \rangle = 9 \left(\frac{C_F \pi}{2} \alpha_s^2 \right) \frac{\left| R'(0) \right|^2}{\pi m^4}$$

Decay in the Singlet Model

$$|H\rangle = \int \frac{d^3k}{(2\pi)^3}\,\Phi^{ij}_{uv}(\mathbf{k})|Q(\mathbf{k})^{iu}Q(-\mathbf{k})^{jv}\rangle \qquad \Phi^{ij}_{uv}(\mathbf{x}) \sim (\cdots)R(x)$$

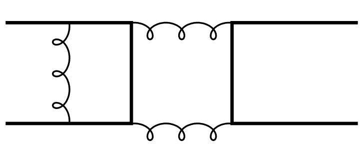
$$\Gamma(\chi_0 \to \mathrm{LH}) \simeq \langle H|2\ \mathrm{Im}$$



$$|H\rangle$$

- one loop

2 Im



$+ \ldots$

$$\longrightarrow \left((\ldots)\alpha_s^2 + (\ldots)\alpha_s^3 \ln\frac{\mu}{m}\right)\frac{\mathbf{S}\cdot\boldsymbol{\nabla}\delta(\mathbf{r})\mathbf{S}\cdot\boldsymbol{\nabla}}{m^4}$$

`Barbieri et al. 76, 79, 80, 81`

Octets

$$|H\rangle \quad = \quad \left(|(Q\bar{Q})_1\rangle + |(Q\bar{Q})_8 g\rangle + \cdots \right) \otimes |nljs\rangle$$

$$\mathcal{O}(1) \qquad\qquad \mathcal{O}(v)$$

$$\psi^\dagger K^{(n)} \chi \chi^\dagger K'^{(n)} \psi = \begin{cases} O_1({}^{2S+1}L_J) \\ O_8({}^{2S+1}L_J) \end{cases}$$

$$\psi^\dagger T^a \chi \chi^\dagger T^a \psi = O_8({}^1S_0) \qquad \psi^\dagger \mathbf{D} \chi \chi^\dagger \mathbf{D} \psi = O_1({}^1P_1) \qquad \ldots$$

$$\mathcal{O}(1) \qquad\qquad \mathcal{O}(v^2)$$

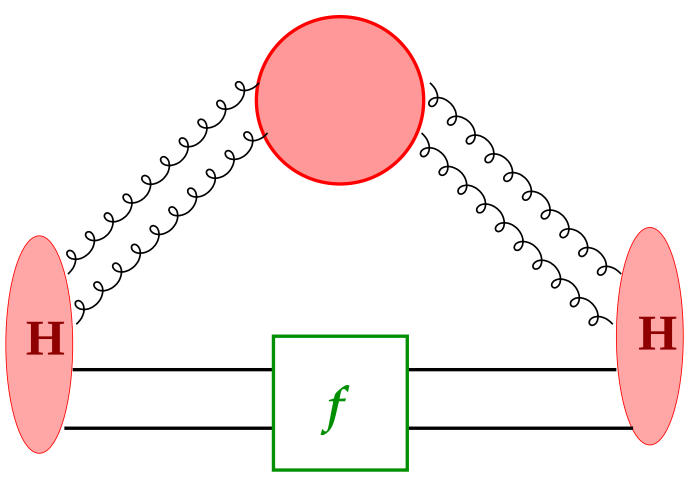
Decay in NRQCD

$$\Gamma(H \to \mathrm{LH}) = -2\,\mathrm{Im}\,\langle H|\,\mathcal{H}\,|H\rangle$$

$$= \sum_n \frac{2\,\mathrm{Im}\, f^{(n)}}{m^{d_n-4}} \langle H|\psi^\dagger K^{(n)} \chi\chi^\dagger K'^{(n)} \psi|H\rangle$$

$$\Gamma(H \to \mathrm{EM}) = \sum_n \frac{2\,\mathrm{Im}\, f^{(n)}_{\mathrm{em}}}{m^{d_n-4}} \langle H|\psi^\dagger K^{(n)} \chi|\mathrm{vac}\rangle\langle \mathrm{vac}|\chi^\dagger K'^{(n)} \psi|H\rangle$$

Bodwin et al. 95



P-wave decays at $\mathcal{O}(mv^5)$

$$\Gamma(\chi_J \to \mathrm{LH}) \;=\; 9\,\mathrm{Im}\,f_1\,\frac{\left|R'(0)\right|^2}{\pi m^4} \;+\; \frac{2\,\mathrm{Im}\,f_8}{m^2}\,\langle\chi|O_8({}^1S_0)|\chi\rangle$$

$$\Gamma(\chi_J \to \gamma\gamma) \;=\; 9\,\mathrm{Im}\,f_{\gamma\gamma}\,\frac{\left|R'(0)\right|^2}{\pi m^4} \qquad\qquad J = 0, 2$$

* Octet and singlet contribute to the same order.

$\Rightarrow$ *The IR divergences of* $\mathrm{Im}\,f_1$ *are absorbed into the non-perturbative operator* $\langle\chi|O_8({}^1S_0)|\chi\rangle$.

Bodwin Braaten Lepage 92

P-wave decays at $\mathcal{O}(mv^5)$

$$\Gamma(\chi_J \to \mathrm{LH}) \;=\; 9\ \mathrm{Im}\, f_1 \,\frac{\left|R'(0)\right|^2}{\pi m^4} \;+\; \frac{2\ \mathrm{Im}\, f_8}{m^2}\, \langle \chi | O_8(^1S_0) | \chi \rangle$$

$$\Gamma(\chi_J \to \gamma\gamma) \;=\; 9\ \mathrm{Im}\, f_{\gamma\gamma} \,\frac{\left|R'(0)\right|^2}{\pi m^4} \qquad\qquad J = 0, 2$$

* *Bottomonium and charmonium (below threshold) P-wave decays depend on 6 non-perturbative parameters [3 w.f. $+$ 3 octet].*

S-wave decays at $\mathcal{O}(mv^5)$

$$\Gamma(V(nS) \to \mathrm{LH}) = \frac{2}{m^2}\Bigg(\mathrm{Im}\, f_1(^3S_1)\langle V|O_1(^3S_1)|V\rangle$$

$$+\mathrm{Im}\, f_8(^3S_1)\langle P|O_8(^1S_0)|P\rangle + \frac{\mathrm{Im}\, f_8(^1S_0)}{3}\langle P|O_8(^3S_1)|P\rangle$$

$$+\mathrm{Im}\, g_1(^3S_1)\frac{\langle P|\mathcal{P}_1(^1S_0)|P\rangle}{m^2} + \frac{\sum_J (2J+1)\mathrm{Im}\, f_8(^3P_J)}{3}\frac{\langle P|O_8(^1P_1)|P\rangle}{m^2}\Bigg)$$

$$\Gamma(P(nS) \to \mathrm{LH}) = \frac{2}{m^2}\Bigg(\mathrm{Im}\, f_1(^1S_0)\langle P|O_1(^1S_0)|P\rangle$$

$$+\mathrm{Im}\, f_8(^1S_0)\langle P|O_8(^1S_0)|P\rangle + \frac{\mathrm{Im}\, f_8(^3S_1)}{3}\langle P|O_8(^3S_1)|P\rangle$$

$$+\mathrm{Im}\, g_1(^1S_0)\frac{\langle P|\mathcal{P}_1(^1S_0)|P\rangle}{m^2} + \mathrm{Im}\, f_8(^1P_1)\frac{\langle P|O_8(^1P_1)|P\rangle}{m^2}\Bigg)$$

* *Bottomonium and charmonium (below threshold) S-wave decays depend on 30 non-perturbative parameters.*

S-wave e.m. decays at $\mathcal{O}(mv^5)$

$$\Gamma(V(nS) \to e^+e^-) = \frac{2}{m^2}\left(\mathrm{Im}\, f_{ee}(^3S_1)\langle V|O_{\mathrm{EM}}(^3S_1)|V\rangle\right.$$

$$\left. +\mathrm{Im}\, g_{ee}(^3S_1)\frac{\langle P|\mathcal{P}_1(^1S_0)|P\rangle}{m^2}\right)$$

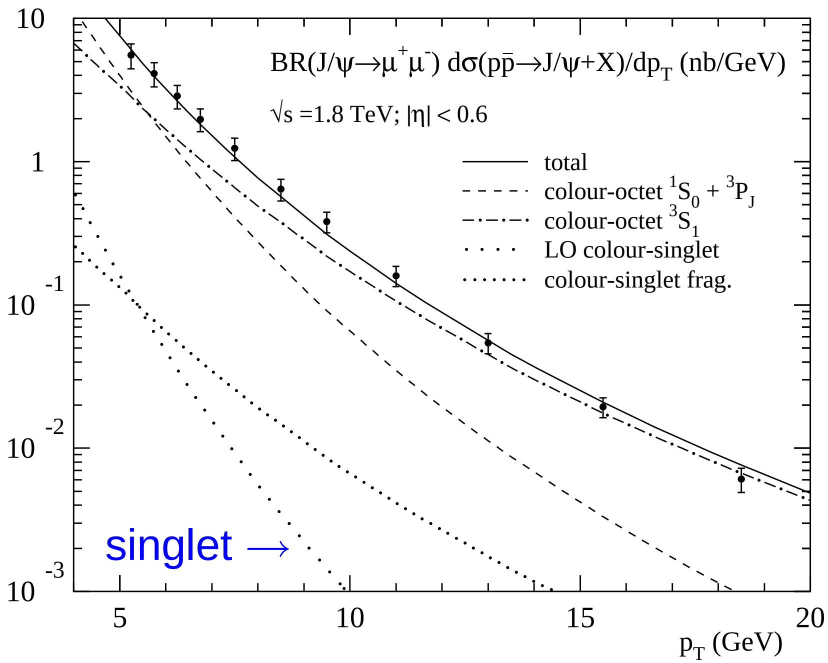
$$\Gamma(P(nS) \to \gamma\gamma) = \frac{2}{m^2}\left(\mathrm{Im}\, f_{\gamma\gamma}(^1S_0)\langle P|O_{\mathrm{EM}}(^1S_0)|P\rangle\right.$$

$$\left. +\mathrm{Im}\, g_{\gamma\gamma}(^1S_0)\frac{\langle P|\mathcal{P}_1(^1S_0)|P\rangle}{m^2}\right)$$

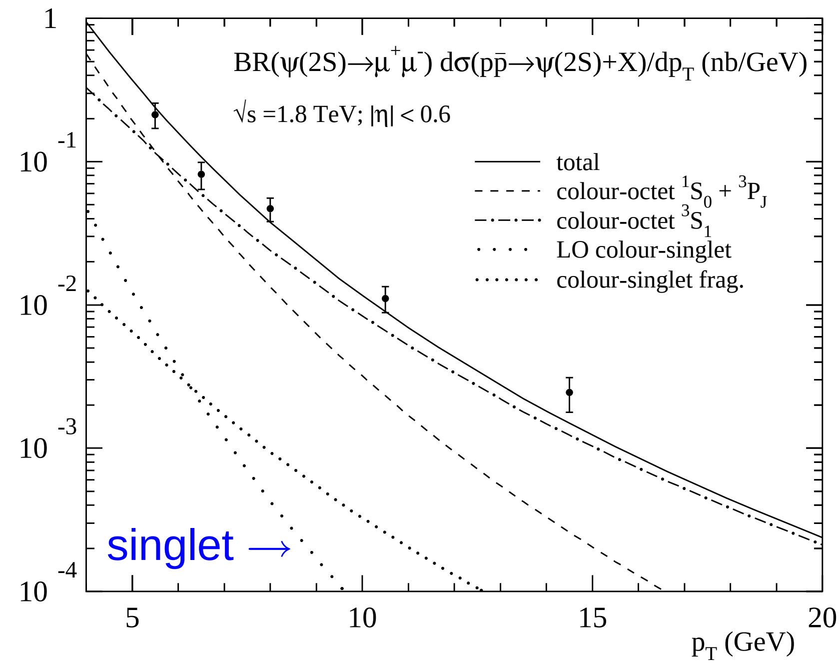
* *Bottomonium and charmonium (below threshold) S-wave e.m. decays depend on 10 extra non-perturbative parameters.*

Quarkonium Production

Octet contributions dominate in production at high p_T.



$pp \rightarrow J/\psi + X$



$pp \rightarrow \psi(2S) + X$

Krämer 01, CDF 97

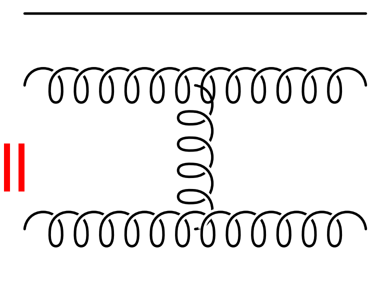
pNRQCD

The Static Spectrum

Gluonic excitations between static quarks are of 3 types:

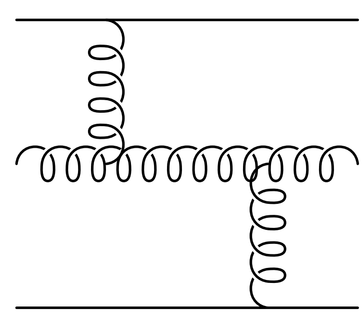
$(Q\bar{Q})_1$

$(Q\bar{Q})_1 +$ Glueball

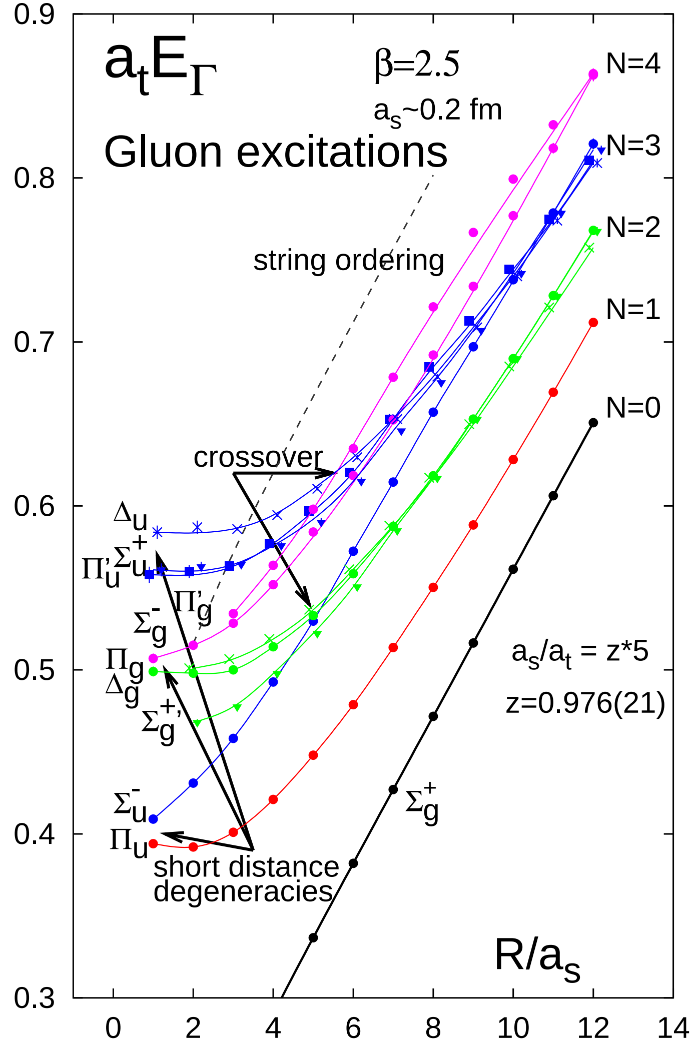


Hybrid

$(Q\bar{Q})_8 G$



Hybrid Static Energies



Juge, Kuti, Morningstar 98, 02

Hybrids and Gluelumps

At short distance, $1/r \gg \Lambda_{\rm QCD}$ the EFT is pNRQCD:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a}$$
$$+\mathrm{Tr}\left\{ \mathrm{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) \mathrm{S} + \mathrm{O}^\dagger \left(iD_0 - \frac{\mathbf{p}^2}{m} - V_o \right) \mathrm{O} \right\}$$
$$+\mathrm{Tr}\left\{ \mathrm{O}^\dagger \mathbf{r}\cdot g\mathbf{E}\, \mathrm{S} + \mathrm{S}^\dagger \mathbf{r}\cdot g\mathbf{E}\, \mathrm{O} + \frac{\mathrm{O}^\dagger \mathbf{r}\cdot g\mathbf{E}\, \mathrm{O}}{2} + \frac{\mathrm{O}^\dagger \mathrm{O} \mathbf{r}\cdot g\mathbf{E}}{2} \right\}$$

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- *At lowest order in the multipole expansion, the singlet decouples while the octet is still coupled to gluons.*

Hybrids and Gluelumps

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- *Static hybrids at short distance are called gluelumps and are described by a static adjoint source (O) in the presence of a gluonic field (H):*

$$\mathrm{H}(R, r, t) = \mathrm{Tr}\{\mathrm{O} H\}$$

Hybrids and Gluelumps

At short distance, $1/r \gg \Lambda_{\rm QCD}$ the EFT is pNRQCD:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a}$$
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- *Depending on the glue operator H and its symmetries, the operator $\mathrm{Tr}\{\mathrm{O}H\}$ describes a specific gluelump of energy E_H.*

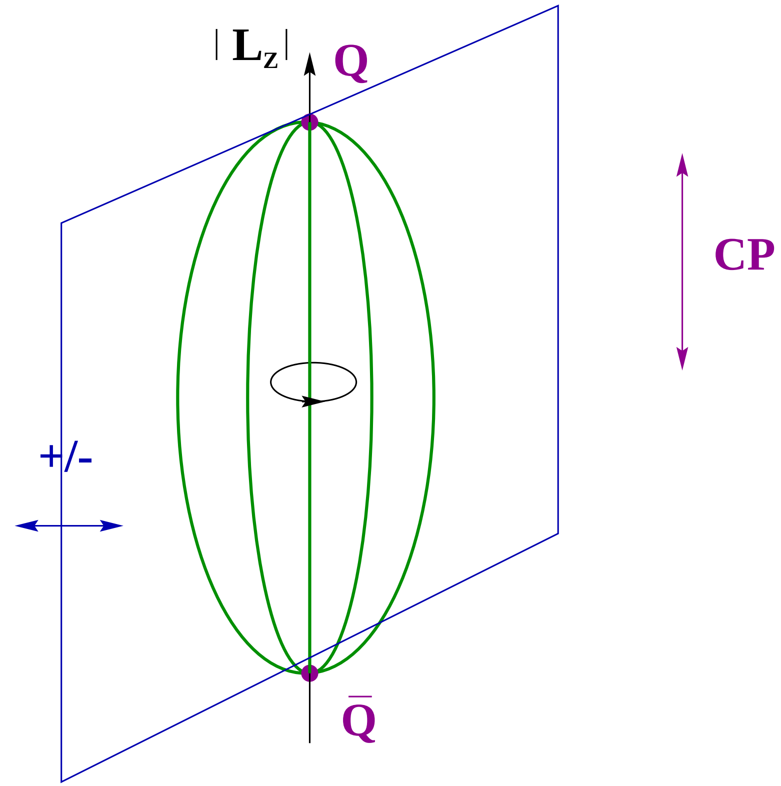
Hybrids and Gluelumps

Symmetries of a
diatomic molecule
+ C.C.

a) $|L_z| = 0, 1, 2, \ldots$
$= \Sigma, \Pi, \Delta \ldots$

b) CP (u/g)

c) Reflection ($+/-$)
(for Σ only)



Hybrids and Gluelumps

Symmetries of a diatomic molecule + C.C.

a) $|L_z| = 0, 1, 2, \ldots$
$= \Sigma, \Pi, \Delta \ldots$

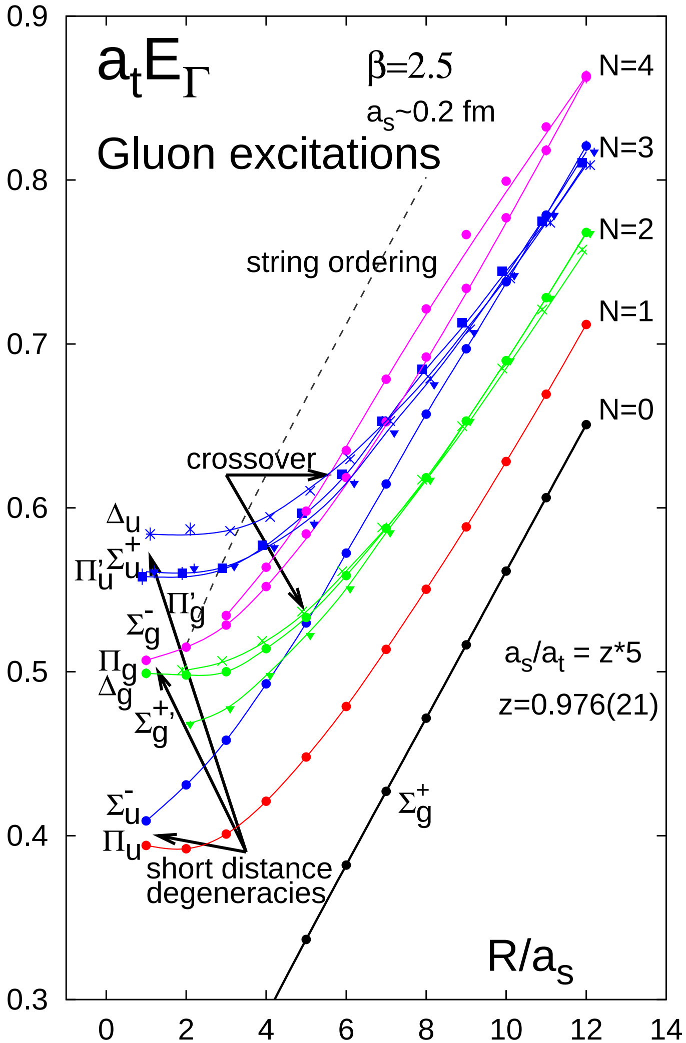
b) CP (u/g)

c) Reflection ($+/-$)
(for Σ only)

	$L = 1$	$L = 2$
$\Sigma_g^{+\,\prime}$	$\mathbf{r} \cdot (\mathbf{D} \times \mathbf{B})$	
Σ_g^-		$(\mathbf{r} \cdot \mathbf{D})(\mathbf{r} \cdot \mathbf{B})$
Π_g	$\mathbf{r} \times (\mathbf{D} \times \mathbf{B})$	
Π_g'		$\mathbf{r} \times ((\mathbf{r} \cdot \mathbf{D})\mathbf{B} + \mathbf{D}(\mathbf{r} \cdot \mathbf{B}))$
Δ_g		$(\mathbf{r} \times \mathbf{D})^i (\mathbf{r} \times \mathbf{B})^j + +(\mathbf{r} \times \mathbf{D})^j (\mathbf{r} \times \mathbf{B})^i$
Σ_u^+		$(\mathbf{r} \cdot \mathbf{D})(\mathbf{r} \cdot \mathbf{E})$
Σ_u^-	$\mathbf{r} \cdot \mathbf{B}$	
Π_u	$\mathbf{r} \times \mathbf{B}$	
Π_u'		$\mathbf{r} \times ((\mathbf{r} \cdot \mathbf{D})\mathbf{E} + \mathbf{D}(\mathbf{r} \cdot \mathbf{E}))$
Δ_u		$(\mathbf{r} \times \mathbf{D})^i (\mathbf{r} \times \mathbf{E})^j + +(\mathbf{r} \times \mathbf{D})^j (\mathbf{r} \times \mathbf{E})^i$

Brambilla Pineda Soto Vairo 99

Hybrids and Gluelumps



	$L=1$	$L=2$
$\Sigma_g^{+\prime}$	$\mathbf{r}\cdot(\mathbf{D}\times\mathbf{B})$	
Σ_g^-		$(\mathbf{r}\cdot\mathbf{D})(\mathbf{r}\cdot\mathbf{B})$
Π_g	$\mathbf{r}\times(\mathbf{D}\times\mathbf{B})$	
Π'_g		$\mathbf{r}\times((\mathbf{r}\cdot\mathbf{D})\mathbf{B}+\mathbf{D}(\mathbf{r}\cdot\mathbf{B}))$
Δ_g		$(\mathbf{r}\times\mathbf{D})^i(\mathbf{r}\times\mathbf{B})^j + {}+(\mathbf{r}\times\mathbf{D})^j(\mathbf{r}\times\mathbf{B})^i$
Σ_u^+		$(\mathbf{r}\cdot\mathbf{D})(\mathbf{r}\cdot\mathbf{E})$
Σ_u^-	$\mathbf{r}\cdot\mathbf{B}$	
Π_u	$\mathbf{r}\times\mathbf{B}$	
Π'_u		$\mathbf{r}\times((\mathbf{r}\cdot\mathbf{D})\mathbf{E}+\mathbf{D}(\mathbf{r}\cdot\mathbf{E}))$
Δ_u		$(\mathbf{r}\times\mathbf{D})^i(\mathbf{r}\times\mathbf{E})^j + {}+(\mathbf{r}\times\mathbf{D})^j(\mathbf{r}\times\mathbf{E})^i$

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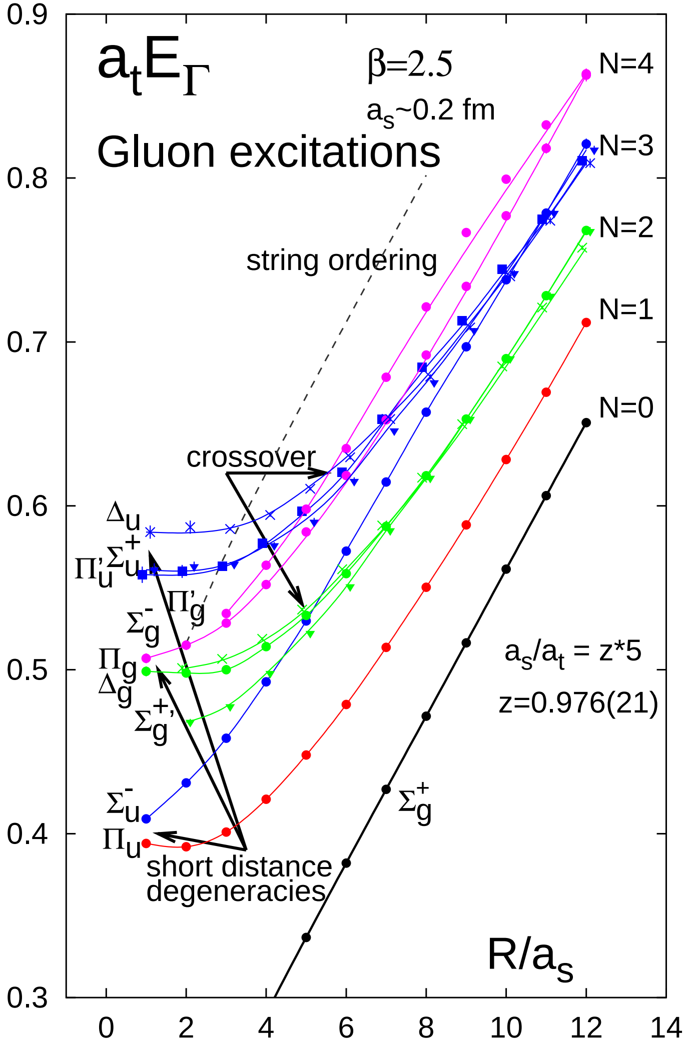
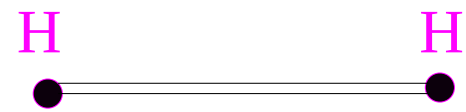
Brambilla Pineda Soto Vairo 99

Hybrids and Gluelumps



Juge Kuti Morningstar 98, 02

At LO in the multipole expansion



$$= e^{-iTE_H}$$

$$E_H = V_o + \frac{i}{T}\ln\langle H^a(\frac{T}{2})\phi^{\rm adj}_{ab}H^b(-\frac{T}{2})\rangle$$

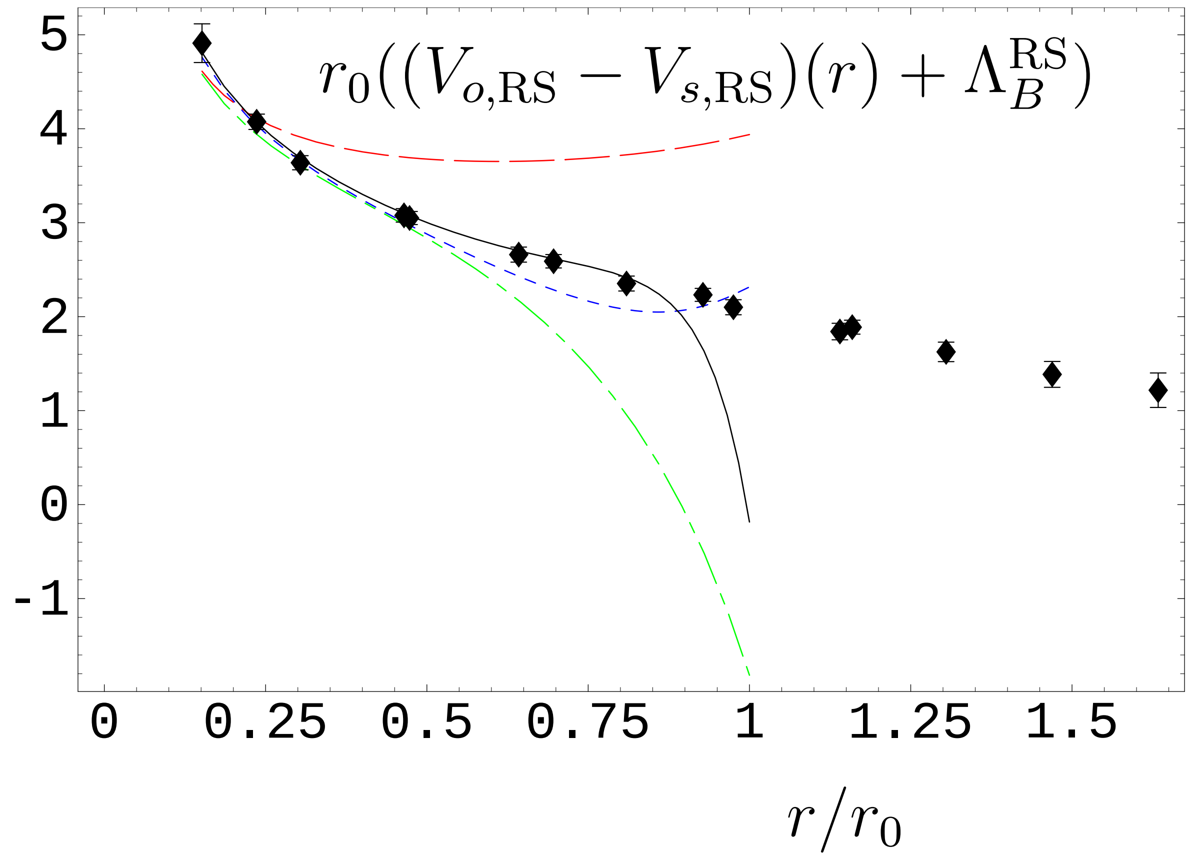
From

$$\langle H^a(\frac{T}{2})\phi^{\rm adj}_{ab}H^b(-\frac{T}{2})\rangle^{\rm np} \sim h\, e^{-iT\Lambda_H}$$

$$\boxed{E_H(r) = V_o(r) + \Lambda_H}$$

Octet potential vs lattice QCD

Renormalon subtraction (RS) is crucial in comparing the perturbative static octet potential with lattice data.



NNLL + 3 loop est.

NNLO

NLO

LO

$\alpha_\mathrm{s} = \alpha_\mathrm{s}(1/r)$

$\nu_f = \nu_{us} = 2.5\, r_0^{-1}$

Lattice data of $E_{\Pi_u} - E_{\Sigma_g^+}$

Bali Pineda 03

Octet potential vs lattice QCD

Λ_B correlation length

$$\Lambda_B^{\rm RS}(\nu_f = 2.5\, r_0^{-1}) = [2.25 \pm 0.10(\text{latt.}) \pm 0.21(\text{th.}) \pm 0.08(\Lambda_{\overline{\rm MS}})]\; r_0^{-1}$$

for $\nu_f = 2.5\, r_0^{-1} \approx 1\ \text{GeV}$

$$\Lambda_B^{\rm RS}(1\,\text{GeV}) = [0.887 \pm 0.039(\text{latt.}) \pm 0.083(\text{th.}) \pm 0.032(\Lambda_{\overline{\rm MS}})]\,\text{GeV}$$

Octet potential vs lattice QCD

Higher Gluelump excitations

J^{PC}	H	$\Lambda_H^{\mathrm{RS}} r_0$	Λ_H^{RS}/GeV
1^{+-}	B_i	2.25(39)	0.87(15)
1^{--}	E_i	3.18(41)	1.25(16)
2^{--}	$D_{\{i}B_{j\}}$	3.69(42)	1.45(17)
2^{+-}	$D_{\{i}E_{j\}}$	4.72(48)	1.86(19)
3^{+-}	$D_{\{i}D_jB_{k\}}$	4.72(45)	1.86(18)
0^{++}	$\mathbf{B}^2$	5.02(46)	1.98(18)
4^{--}	$D_{\{i}D_jD_kB_{l\}}$	5.41(46)	2.13(18)
1^{-+}	$(\mathbf{B} \wedge \mathbf{E})_i$	5.45(51)	2.15(20)

Foster Michael 99, Bali Pineda 03

pNRQCD

$$mv \sim \Lambda_{\rm QCD}$$

pNRQCD for $mv \sim \Lambda_{\mathrm{QCD}}$

- All scales above mv^2 are integrated out.

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- All gluonic excitations between heavy quarks are integrated out since they develop a gap of order $\Lambda_{\rm QCD}$ with the static $Q\bar{Q}$ energy.

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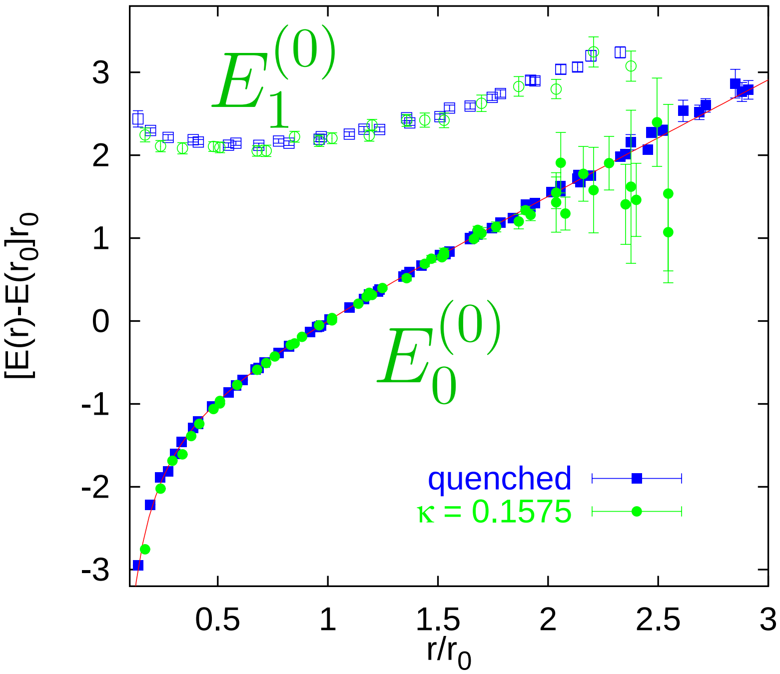

Bali et al. 98

($r_0 \simeq 0.5\,$fm)

pNRQCD for $mv \sim \Lambda_{\rm QCD}$

- All scales above mv^2 are integrated out.
- All gluonic excitations between heavy quarks are integrated out since they develop a gap of order $\Lambda_{\rm QCD}$ with the static $Q\bar{Q}$ energy.

$\Rightarrow$ The singlet quarkonium field S of energy mv^2 is the only the degree of freedom of pNRQCD (up to ultrasoft light quarks, e.g. pions).

pNRQCD for $mv \sim \Lambda_{\mathrm{QCD}}$

$$\mathcal{L} = \mathrm{Tr}\left\{ \mathrm{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) \mathrm{S} \right\}$$

Brambilla Pineda Soto Vairo 00

pNRQCD for $mv \sim \Lambda_{\rm QCD}$

$$\mathcal{L} = \mathrm{Tr}\left\{ \mathrm{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) \mathrm{S} \right\}$$

Brambilla Pineda Soto Vairo 00

- The potential V_s $(\mathrm{Re}\, V_s + i\, \mathrm{Im}\, V_s)$ is non-perturbative:
  (a) to be determined from the lattice;
  (b) to be determined from QCD vacuum models.

Creutz et al. 82, Campostrini 85, Michael 85, Born et al. 94,
Bali Schilling Wachter 97, Brambilla et al. 93, 95, 97, 98

The Quantum-Mechanical Matching

The matching condition is:

$$\langle H | \mathcal{H} | H \rangle = \langle nljs | \frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}}{m^n} | nljs \rangle$$

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$$\mathcal{H} = \mathcal{H}^{(0)} + \frac{\delta\mathcal{H}^{(1)}}{m} + \frac{\delta\mathcal{H}^{(2)}}{m^2} + \frac{\delta\mathcal{H}^{(3)}}{m^3} + \frac{\delta\mathcal{H}^{(4)}}{m^4} + \dots$$

$$\mathcal{H}^{(0)} = \int d^3\mathbf{x} \, \frac{1}{2} \left(\mathbf{\Pi}^a \mathbf{\Pi}^a + \mathbf{B}^a \mathbf{B}^a \right) - \sum^{n_f} \bar{q} \, i\mathbf{D} \cdot \boldsymbol{\gamma} \, q$$

$$\delta\mathcal{H}^{(1)} = - \int d^3\mathbf{x} \, \psi^\dagger \left(\frac{\mathbf{D}^2}{2} + c_F \, g \, \mathbf{S} \cdot \mathbf{B} \right) \psi + \text{antip.}$$

$\dots \qquad \dots \qquad \dots$

The Quantum-Mechanical Matching

The matching condition is:

$$\langle H | \mathcal{H} | H \rangle = \langle nljs | \frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}}{m^n} | nljs \rangle$$

In a QM language:

$$\mathcal{H}^{(0)} |\underline{n}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)} = E_n^{(0)}(\mathbf{x}_1, \mathbf{x}_2) |\underline{n}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)}$$

$$|\underline{n}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)} = \psi^\dagger(\mathbf{x}_1)\chi(\mathbf{x}_2) |n; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)}$$

$\mathbf{x}_j$ are the quark positions n : $CP, \ldots$

$|\underline{0}\rangle^{(0)} = |(Q\bar{Q})_1\rangle \to$ Quarkonium Singlet

$|\underline{n > 0}\rangle^{(0)} = |(Q\bar{Q}) g^{(n)}\rangle \to$ Higher Gluonic Excitations

The Quantum-Mechanical Matching

The matching condition is:

$$\langle H | \mathcal{H} | H \rangle = \langle nljs | \frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}}{m^n} | nljs \rangle$$

Expanding in $1/m$:

$$|\underline{0}; \mathbf{x}_1, \mathbf{x}_2\rangle = |\underline{0}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)} + \sum_{n \neq 0} \int d^3 z_1 \, d^3 z_2 |\underline{n}; \mathbf{z}_1, \mathbf{z}_2\rangle^{(0)}$$

$$\times \frac{{}^{(0)}\langle \underline{n}; \mathbf{z}_1, \mathbf{z}_2 | \delta\mathcal{H}^{(1)} | \underline{0}; \mathbf{x}_1, \mathbf{x}_2 \rangle^{(0)}}{E_0^{(0)}(z) - E_n^{(0)}(x)} + \ldots$$

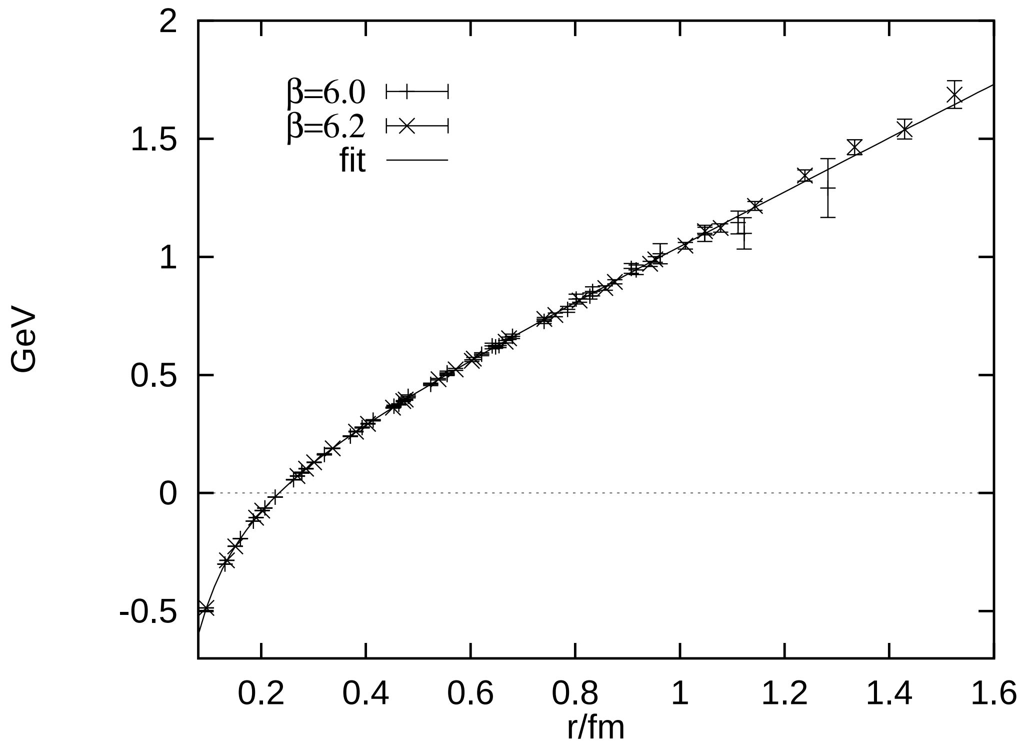
$$|H\rangle \to |\underline{0}; \mathbf{x}_1, \mathbf{x}_2\rangle \otimes |nljs\rangle$$

The non-perturbative Potentials

$$V_s = {V_s}^{(0)} + \frac{{V_s}^{(1)}}{m} + \frac{{V_s}^{(2)}}{m^2} + \ldots$$

The non-perturbative Potentials

$$V_s^{(0)} = \lim_{T\to\infty} \frac{i}{T} \ln\langle W(r\times T)\rangle = \lim_{T\to\infty} \frac{i}{T} \ln \langle \boxed{} \rangle$$



Bali Schilling Wachter 97

The non-perturbative Potentials

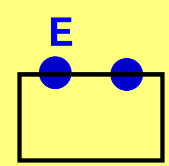
$$V_s^{(1)} = -{}^{(0)}\langle 0|\mathbf{D}^2|0\rangle^{(0)} = -\boldsymbol{\nabla}^2 + \sum_{k\neq 0}\left|\frac{{}^{(0)}\langle k|g\mathbf{E}|0\rangle^{(0)}}{E_0^{(0)} - E_k^{(0)}}\right|^2$$

kinetic energy

Since

$$\langle\langle \mathbf{E}(t)\cdot\mathbf{E}(0)\rangle\rangle_\Box = \sum_k |{}^{(0)}\langle 0|g\mathbf{E}|k\rangle^{(0)}|^2 e^{-iE_0^{(0)}T - i(E_k^{(0)} - E_0^{(0)})t}$$

$$V_s^{(1)} = -\frac{1}{2}\int_0^\infty dt\, t \;\langle \quad \rangle$$



Brambilla Pineda Soto Vairo 00

The non-perturbative Potentials

$$V_{\rm SD}^{(2)} = \frac{1}{r}\left(c_F \epsilon^{kij}\, \frac{2r^k}{r} i \int_0^\infty dt\, t \,\left\langle \overset{E}{\underset{B}{\boxed{\;i \quad j\;}}} \right\rangle - \frac{1}{2} V_s^{(0)\prime} \right) (\mathbf{S}_1 + \mathbf{S}_2)\cdot \mathbf{L}$$

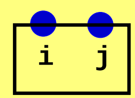
$$- {c_F}^2 \hat{r}_i \hat{r}_j i \int_0^\infty dt \left(\left\langle \boxed{\;i \quad j\;} \right\rangle - \frac{\delta_{ij}}{3} \left\langle \boxed{\quad\quad} \right\rangle \right)$$

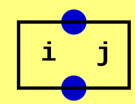
$$\times \left(\mathbf{S}_1 \cdot \mathbf{S_2} - 3(\mathbf{S}_1\cdot \hat{\mathbf{r}})(\mathbf{S_2}\cdot\hat{\mathbf{r}}) \right)$$

$$+ \left(\frac{2}{3} {c_F}^2 i \int_0^\infty dt \left\langle \boxed{\quad\quad} \right\rangle - 4(d_2 + C_F d_4)\delta^{(3)}(\mathbf{r}) \right) \mathbf{S}_1 \cdot \mathbf{S_2}$$

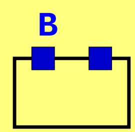
Eichten Feinberg 81, Gromes 84, Chen et al. 95 Brambilla Vairo 99

The non-perturbative Potentials

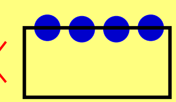
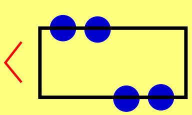
$V^{(2)}_{\rm SI} = p^i \Big(i\int_0^\infty dt\, t^2 \langle$ 

$\rangle + \langle$ 

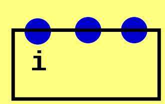
$\rangle \Big) p^j$

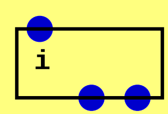
$-\frac{c_F^2}{2} i \int_0^\infty dt \langle$ 

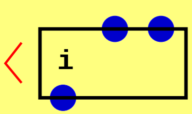
$\rangle + (d_1 + C_F d_3 + \pi C_F \alpha_{\rm s} c_D)\delta^{(3)}(\mathbf{r})$

$-i \int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 (t_2 - t_3)^2 \Big(\langle$  $\rangle + \langle$  $\rangle \Big)$

$+ \int_0^\infty dt_1 \int_0^{t_1} dt_2 (t_1 - t_2)^2 \nabla^i$

$\times \Big(\langle$ 

$\rangle + \frac{1}{2} \langle$ 

$\rangle + \frac{1}{2} \langle$ 

$\rangle \Big)$

$$-2 b_3 f_{abc} \int d^3\mathbf{x}\, g \langle\!\langle G^a_{\mu\nu}(\mathbf{x}) G^b_{\mu\alpha}(\mathbf{x}) G^c_{\nu\alpha}(\mathbf{x}) \rangle\!\rangle^c_\Box$$

Brambilla et al. 88 90, Pineda Vairo 00

Imaginary parts of the Potential

$$\mathrm{Im}\, V_s \Big|_{\mathrm{P-wave}} = \Omega_{ij}^{SJ} \boldsymbol{\nabla}^i \delta^3(\mathbf{r}) \boldsymbol{\nabla}^j$$
$$\times \left[3\, \frac{\mathrm{Im}\, f_1({}^{2S+1}\mathrm{P}_J)}{m^4} + \frac{\mathcal{E}}{27} \frac{\mathrm{Im}\, f_8({}^{2S+1}\mathrm{S}_S)}{m^4} \right]$$

where

$$\mathcal{E} = 18 \sum_{n \neq 0} \frac{\langle 0|g\mathbf{E}|n\rangle \cdot \langle n|g\mathbf{E}|0\rangle}{(E_n^{(0)} - E_0^{(0)})^4}$$
$$= \frac{1}{2} \int_0^\infty dt\, t^3 \left\langle g\mathbf{E}^a(t,\mathbf{0}) \Phi_{ab}(t,0;\mathbf{0}) g\mathbf{E}^b(0,\mathbf{0}) \right\rangle$$

P-wave decays at $\mathcal{O}(mv^5)$

$$\Gamma(\chi_J \to \mathrm{LH}) = \frac{\left|R'(0)\right|^2}{\pi m^4} \left[9 \,\mathrm{Im}\, f_1 \; + \; \frac{\mathrm{Im}\, f_8}{9} \, \mathcal{E} \right]$$

$$\Gamma(\chi_J \to \gamma\gamma) \; = \; 9 \,\mathrm{Im}\, f_{\gamma\gamma} \, \frac{\left|R'(0)\right|^2}{\pi m^4} \quad J = 0, 2$$

Brambilla et al. 01, 02, 03

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Brambilla et al. 01, 02, 03

* $\langle\chi|O_8({}^1S_0)|\chi\rangle = \frac{\left|R'(0)\right|^2}{18\pi m^2}\,\mathcal{E}; \quad \mathcal{E} \equiv \int_0^\infty dt\, t^3\, \langle \mathrm{Tr}(g\mathbf{E}(t)\, g\mathbf{E}(0))\rangle$

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Brambilla et al. 01, 02, 03

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* *The quarkonium state dependence factorizes.*

P-wave decays at $\mathcal{O}(mv^5)$

$$\Gamma(\chi_J \to \mathrm{LH}) = \frac{\left|R'(0)\right|^2}{\pi m^4} \left[9 \,\mathrm{Im}\, f_1 \; + \; \frac{\mathrm{Im}\, f_8}{9}\, \mathcal{E} \right]$$

$$\Gamma(\chi_J \to \gamma\gamma) \; = \; 9 \,\mathrm{Im}\, f_{\gamma\gamma}\, \frac{\left|R'(0)\right|^2}{\pi m^4} \quad J = 0, 2$$

Brambilla et al. 01, 02, 03

* $\langle \chi | O_8(^1S_0) | \chi \rangle = \frac{\left|R'(0)\right|^2}{18\pi m^2}\, \mathcal{E}; \quad \mathcal{E} \equiv \int_0^\infty dt\, t^3 \, \langle \mathrm{Tr}(g\mathbf{E}(t)\, g\mathbf{E}(0)) \rangle$

* *Bottomonium and charmonium (below threshold) P-wave decays depend on 4 non-perturbative parameters [3 w.f.* $+$ *1 corr.].*

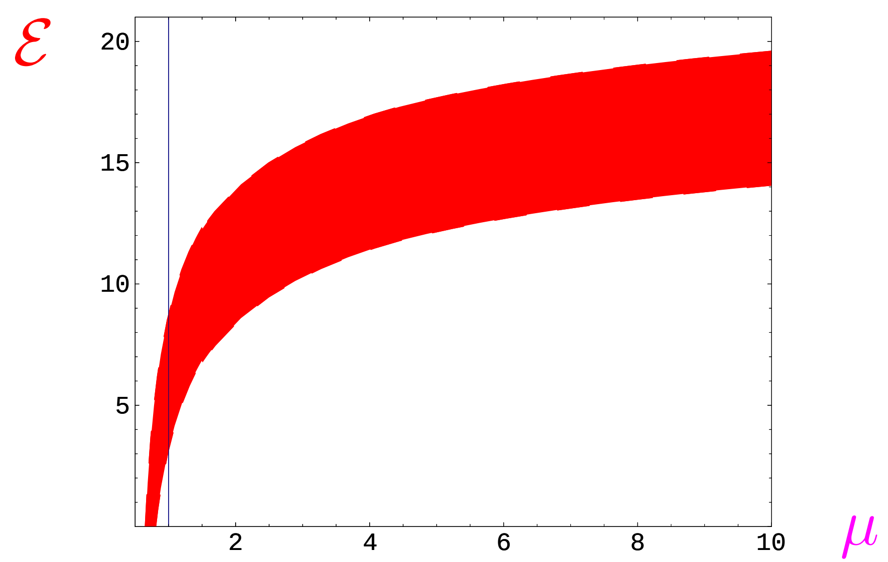
Determination of $\mathcal{E}$ from charmonium data

Process	Γ (MeV)	Reference
$\Gamma(\chi_{c0} \to \mathrm{LH})$	9.7 ± 1.1	E835
$\Gamma(\chi_{c0} \to \mathrm{LH})$	14.3 ± 3.6	BES
$\Gamma(\chi_{c0} \to \mathrm{LH})$	13.5 ± 5.4	CBALL
$\Gamma(\chi_{c1} \to \mathrm{LH})$	0.64 ± 0.12	E760
$\Gamma(\chi_{c2} \to \mathrm{LH})$	1.71 ± 0.18	E760

From the charmonium decay-width ratios we get:

$$\mathcal{E}(1\,\mathrm{GeV}) = 5.3^{+3.5}_{-2.2} \quad \text{[exp]}$$

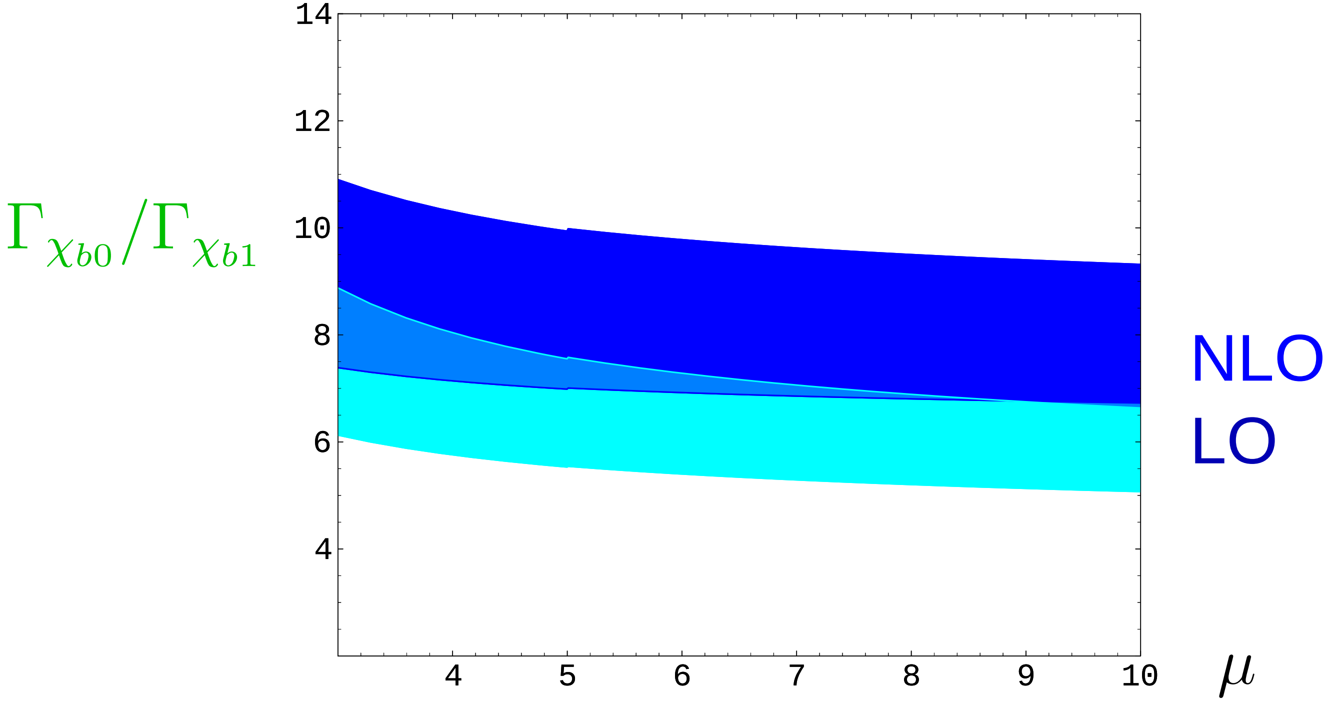
$$\mathcal{E}(\mu) = \mathcal{E}(m) + \frac{96}{\beta_0} \ln \frac{\alpha_s(m)}{\alpha_s(\mu)}$$



Bottomonium P-wave decays

$$\frac{\Gamma(\chi_{b0}(1P) \to \mathrm{LH})}{\Gamma(\chi_{b1}(1P) \to \mathrm{LH})} = \frac{\Gamma(\chi_{b0}(2P) \to \mathrm{LH})}{\Gamma(\chi_{b1}(2P) \to \mathrm{LH})} = 8.0 \pm 1.3 \quad [\mathcal{E}]$$

(CleoIII 02) $= 19.3 \pm 9.8$



Brambilla Eiras Pineda Soto Vairo 01, Vairo 02, CLEO coll. 02

S-wave octet matrix elements

At leading order in the v and $\Lambda_{\rm QCD}/m$ expansion:

$$\langle V|O_8({}^3S_1)|V\rangle = \langle P|O_8({}^1S_0)|P\rangle = C_A\,\frac{|R(0)|^2}{2\pi}\left(-\frac{2(C_A/2-C_F)\mathcal{E}_3^{(2)}}{3m^2}\right)$$

$$\langle V|O_8({}^1S_0)|V\rangle = \frac{\langle P|O_8({}^3S_1)|P\rangle}{3} = C_A\,\frac{|R(0)|^2}{2\pi}\left(-\frac{(C_A/2-C_F)c_F^2\mathcal{B}_1}{3m^2}\right)$$

$$\langle V|O_8({}^3P_J)|V\rangle = \frac{\langle P|O_8({}^1P_1)|P\rangle}{3} = (2J+1)\,C_A\,\frac{|R(0)|^2}{2\pi}\left(-\frac{(C_A/2-C_F)\mathcal{E}_1}{9}\right)$$

$$\langle \chi|O_8({}^1S_0)|\chi\rangle = \frac{T_F}{3}\,\frac{|R'(0)|^2}{\pi m^2}\mathcal{E}_3$$

$$\langle V|\mathcal{P}_1({}^3S_1)|V\rangle = \langle P|\mathcal{P}_1({}^1S_0)|P\rangle = \langle V|\mathcal{P}_{\rm EM}({}^3S_1)|V\rangle$$

$$= \langle P|\mathcal{P}_{\rm EM}({}^1S_0)|P\rangle = C_A\,\frac{|R(0)|^2}{2\pi}\left(mE_{n0}^{(0)} - \mathcal{E}_1\right)$$

S-wave decays

- $$R_n^V \equiv \frac{\Gamma(V(nS) \to LH)}{\Gamma(V(nS) \to e^+e^-)} \qquad R_n^P \equiv \frac{\Gamma(P(nS) \to LH)}{\Gamma(P(nS) \to \gamma\gamma)}$$

It is a prediction of pNRQCD that, for the states for which $\Lambda_{\rm QCD} \gg mv^2$, the wave-function dependence drops out.
$\Big[$ *Residual m dependence in $1/m$, $E_{n0}^{(0)}$ and $\mathrm{Im}\, f$;*
residual n dependence in $E_{n0}^{(0)}$. $\Big]$

S-wave decays

-

$$\frac{R_n^V}{R_m^V} = 1 + \left(\frac{\mathrm{Im}\, g_1(^3S_1)}{\mathrm{Im}\, f_1(^3S_1)} - \frac{\mathrm{Im}\, g_{ee}(^3S_1)}{\mathrm{Im}\, f_{ee}(^3S_1)} \right) \frac{M_n - M_m}{m},$$

$$\frac{R_n^P}{R_m^P} = 1 + \left(\frac{\mathrm{Im}\, g_1(^1S_0)}{\mathrm{Im}\, f_1(^1S_0)} - \frac{\mathrm{Im}\, g_{\gamma\gamma}(^1S_0)}{\mathrm{Im}\, f_{\gamma\gamma}(^1S_0)} \right) \frac{M_n - M_m}{m}.$$

For $m_b = 5$ *GeV,* $R_2^\Upsilon / R_3^\Upsilon \simeq 1.3$ *[pdg* $\simeq 1.4$*]*

$$\mathrm{Im}\, g_1(^1S_0)/\mathrm{Im}\, f_1(^1S_0) - \mathrm{Im}\, g_{\gamma\gamma}(^1S_0)/\mathrm{Im}\, f_{\gamma\gamma}(^1S_0) \sim \alpha_s$$

$\Rightarrow$*up to* $\mathcal{O}(v^3)$*,* R_n^P *is equal for all radial excitations.*

NRQCD power counting & pNRQCD

* In pNRQCD:

$$\langle V|\mathcal{P}_1({}^3S_1)|V\rangle = 3\frac{|R_{n0}^{(0)}|^2}{2\pi}\left(mE_{n0}^{(0)} - \mathcal{E}_1\right)$$

* Using the "standard NRQCD power counting":

Gremm Kapustin 97

$$\langle V_Q(nS)|\mathcal{P}_1({}^3S_1)|V_Q(nS)\rangle_{\mathrm{GK}} = 3\frac{|R_{n0}^{(0)}|^2}{2\pi}mE_{n0}^{(0)}$$

NRQCD power counting & pNRQCD

* In pNRQCD:

$$\langle V|\mathcal{P}_1({}^3S_1)|V\rangle = 3\frac{|R_{n0}^{(0)}|^2}{2\pi}\left(mE_{n0}^{(0)} - \mathcal{E}_1\right)$$

* Using the "standard NRQCD power counting":

Gremm Kapustin 97

$$\langle V_Q(nS)|\mathcal{P}_1({}^3S_1)|V_Q(nS)\rangle_{\rm GK} = 3\frac{|R_{n0}^{(0)}|^2}{2\pi} mE_{n0}^{(0)}$$

The difference clarifies the range of validity of the "standard NRQCD power counting": $E_{n0}^{(0)} \sim mv^2$, $\mathcal{E}_1 \sim \Lambda_{\rm QCD}^2$:

- If $\Lambda_{\rm QCD} \sim mv$, then $\mathcal{E}_1 \sim mE_{n0}^{(0)}$
- If $\Lambda_{\rm QCD} \sim mv^2$, then $\mathcal{E}_1 \ll mE_{n0}^{(0)}$

Higher order matrix elements

$$\langle H|O|H\rangle = \int d^3r \int d^3r' \int d^3R \int d^3R' \, \frac{\langle \mathbf{P}=0|\mathbf{R}\rangle\langle njls|\mathbf{r}\rangle}{\langle \mathbf{P}=0|\mathbf{P}=0\rangle}$$

$$\times \left[\langle \underline{0}; \mathbf{x}_1\mathbf{x}_2| \int d^3\xi \; O(\boldsymbol{\xi}) \; |\underline{0}; \mathbf{x}'_1\mathbf{x}'_2\rangle \right] \langle \mathbf{R}'|\mathbf{P}=0\rangle\langle \mathbf{r}'|njls\rangle$$

Ex. Consider the dimension-9 operator

$$\mathcal{F}_{\mathrm{EM}}(^3P_0) = \frac{1}{6}\psi^\dagger \boldsymbol{\sigma}\cdot g\mathbf{E}\chi|\mathrm{vac}\rangle\langle\mathrm{vac}|\chi^\dagger\boldsymbol{\sigma}\cdot\mathbf{D}\psi + \mathrm{H.c.}$$

that appears in $\chi_{c0} \to \gamma\gamma$ at $\mathcal{O}(mv^7)$. One obtains

$$\langle \chi_Q(n01)|\mathcal{F}_{\mathrm{EM}}(^3P_0)|\chi_Q(n01)\rangle = -3\frac{|R'(0)|^2}{\pi}\frac{\mathcal{E}_1}{m}$$

Conclusions

- Heavy quarkonium in the non-perturbative regime is accessible to a systematic study inside QCD.

  Wave-functions and potentials may be precisely defined in terms of QCD parameters.

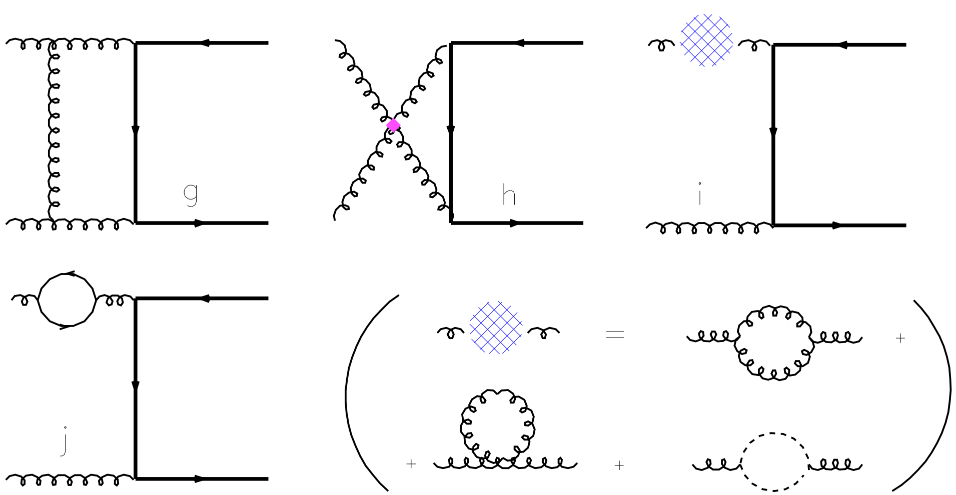
- Accurate determinations interesting both for the phenomenology and for the structure of the QCD vacuum are possible.

- *In lattice calculations for quantities that involves two very different scales* $Q \gg q$ *it should hold*

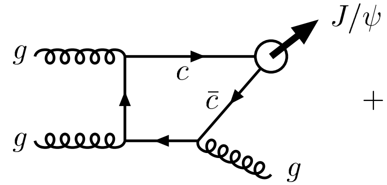
$$L^{-1} \ll q \ll Q \ll a^{-1}$$

$a =$ *lattice spacing*, $L =$ *lattice size*

NLO

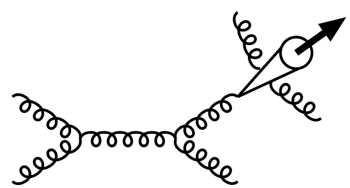


(a) leading-order colour-singlet: $\quad g + g \to c\bar{c}[^3S_1^{(1)}] + g$



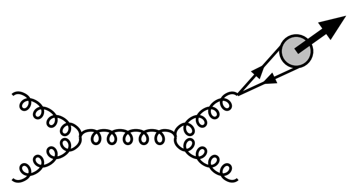
$+ \ldots \qquad \sim\ \alpha_s^3\,\frac{(2m_c)^4}{p_t^8}$

(b) colour-singlet fragmentation: $\quad g + g \to [c\bar{c}[^3S_1^{(1)}] + gg] + g$



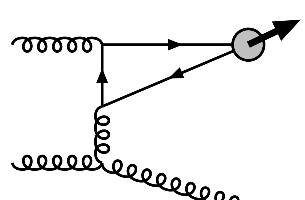
$+ \ldots \qquad \sim\ \alpha_s^5\,\frac{1}{p_t^4}$

(c) colour-octet fragmentation: $\quad g + g \to c\bar{c}[^3S_1^{(8)}] + g$

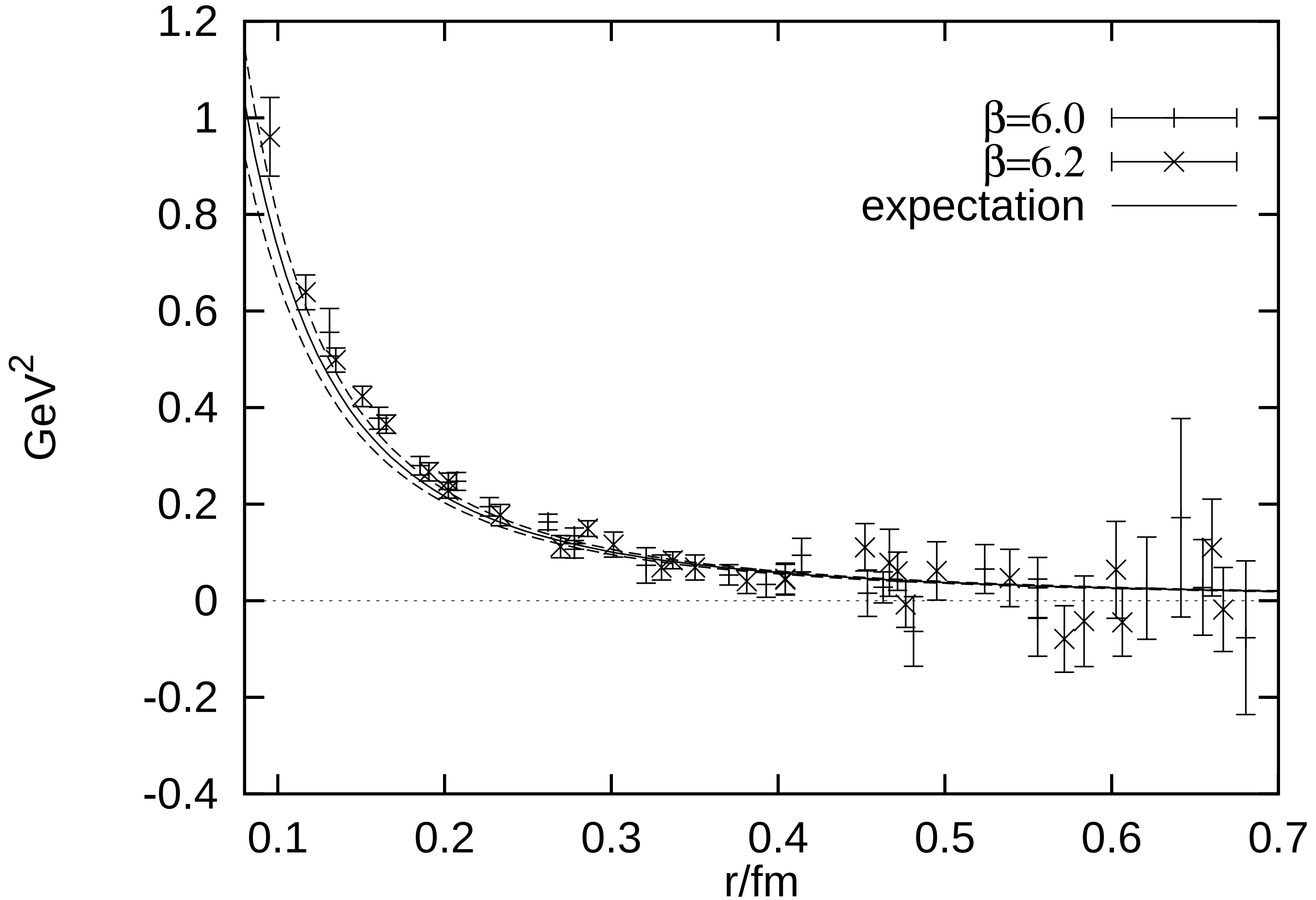


$+ \ldots \qquad \sim\ \alpha_s^3\,\frac{1}{p_t^4}\,v^4$

(d) colour-octet t-channel gluon exchange: $\quad g + g \to c\bar{c}[^1S_0^{(8)}, {}^3P_J^{(8)}] + g$

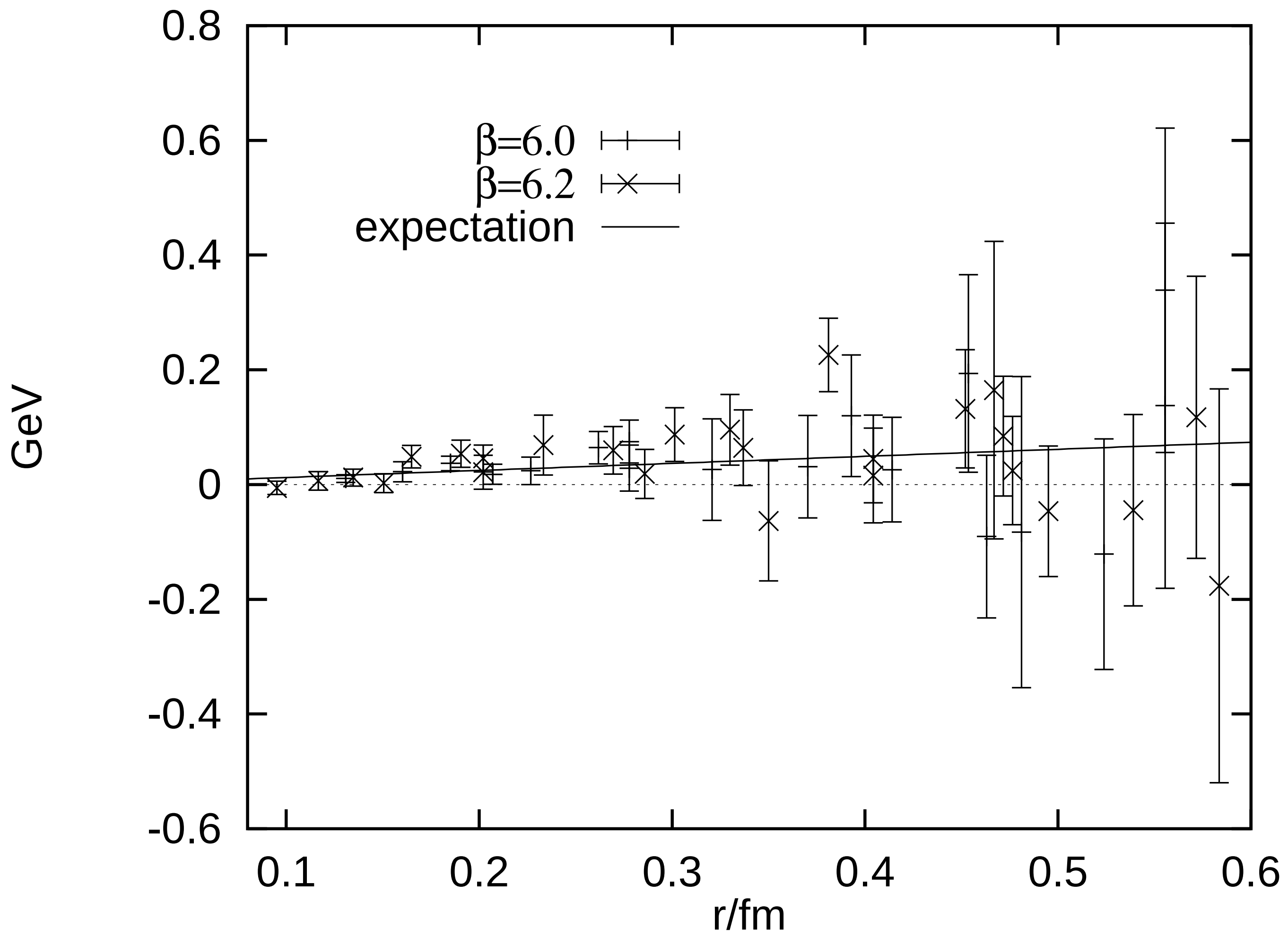


$+ \ldots \qquad \sim\ \alpha_s^3\,\frac{(2m_c)^2}{p_t^6}\,v^4$



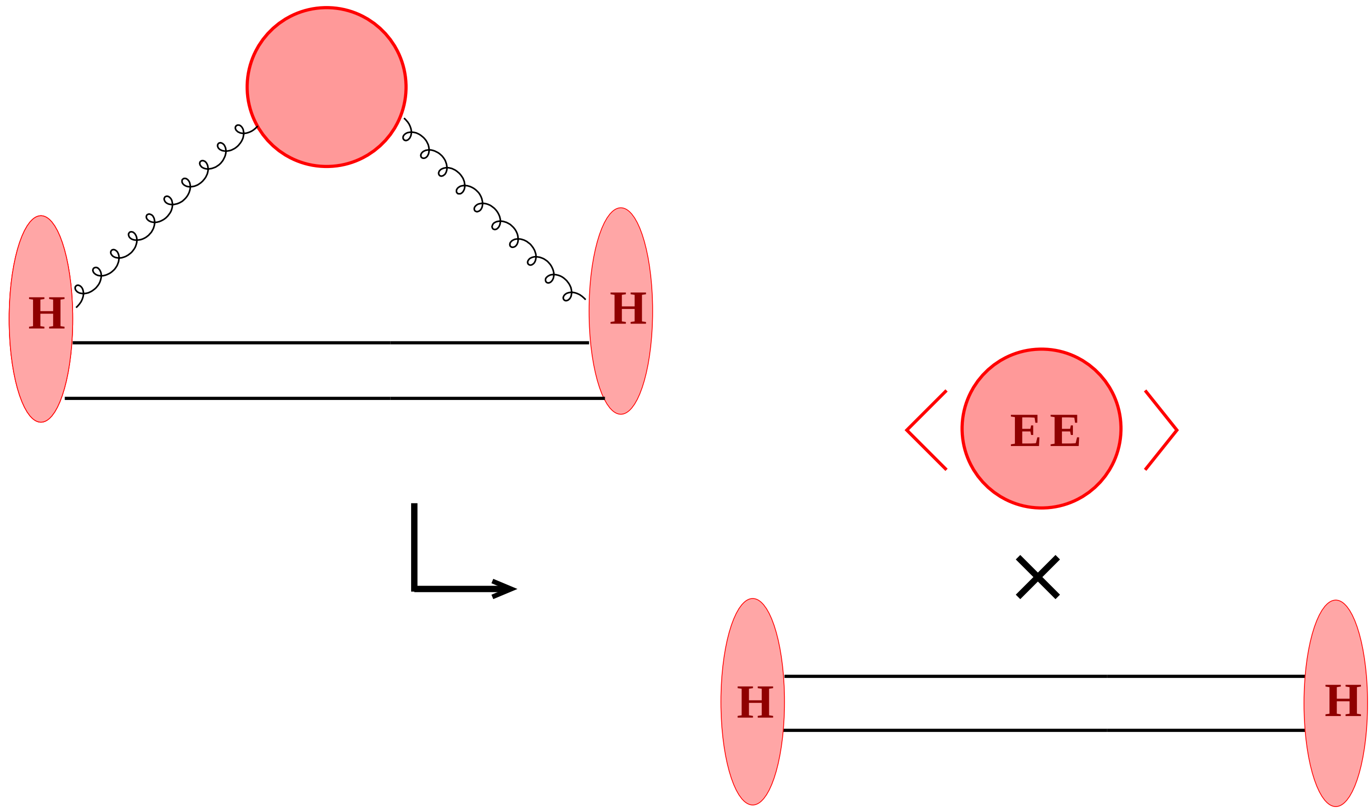
$$\epsilon^{kij}\,\hat{r}^k \int_0^\infty dt\, t \,\langle \boxed{\mathtt{i}\quad \mathtt{j}} \rangle$$

Bali Schilling Wachter 97



$$-\frac{1}{6}\int_0^{\infty} dt\, t^2 \langle \quad \rangle$$

Bali Schilling Wachter 97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$$9\,\mathrm{Im}\,f_1 = \cdots - \frac{8}{9} n_f \alpha_s^3 \ln\frac{\mu}{2m} + \dots$$

$$\frac{\mathrm{Im}\,f_8}{9}\mathcal{E} = \frac{8}{9} n_f \alpha_s^3 \ln\frac{\mu}{\mu_0} + \dots$$

1) $\mathrm{Im}\,f_8 = n_f \dfrac{\pi\alpha_s^2}{6}$

2) $\mathcal{E} \equiv \displaystyle\int_0^\infty dt\, t^3\, \langle \mathrm{Tr}(g\mathbf{E}(t)\, g\mathbf{E}(0))\rangle$

$\mathcal{E} = \displaystyle\int_0^\infty dt\, t^3\, \mathrm{Tr}\{T^a T^a\}\, g^2 \int^\mu \frac{d^3k}{(2\pi)^3} k\, e^{-i\,k\,t} + \dots \qquad \mathrm{Tr}\{T^a T^a\} = 4$

3) $\displaystyle\int_0^\infty dt\, t^3\, e^{-itk} = \frac{6}{k^4}$

4) $\displaystyle\int^\mu \frac{d^3k}{(2\pi)^3} \frac{1}{k^3} \simeq \frac{1}{2\pi^2} \ln\frac{\mu}{\mu_0}$

$$\mathcal{E}_n = \frac{1}{N_c}\int_0^\infty dt\, t^n \langle \mathrm{Tr}(g\mathbf{E}(t)\cdot g\mathbf{E}(0))\rangle$$

$$\mathcal{B}_n = \frac{1}{N_c}\int_0^\infty dt\, t^n \langle \mathrm{Tr}(g\mathbf{B}(t)\cdot g\mathbf{B}(0))\rangle$$

$$\begin{aligned}\mathcal{E}_3^{(2)} \quad = \quad & \frac{1}{4N_c}\int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3\, (t_2 - t_3)^3 \Big\{ \langle \mathrm{Tr}(\{g\mathbf{E}(t_1)\cdot, g\mathbf{E}(t_2)\}\,\{g\mathbf{E}(t_3)\cdot, g\mathbf{E}(0)\})\rangle_c \\ & - \frac{4}{N_c}\langle \mathrm{Tr}(g\mathbf{E}(t_1)\cdot g\mathbf{E}(t_2))\,\mathrm{Tr}(g\mathbf{E}(t_3)\cdot g\mathbf{E}(0))\rangle_c \Big\}\end{aligned}$$

where

$$\begin{aligned}\langle\{g\mathbf{E}(t_1)\cdot, g\mathbf{E}(t_2)\}\,\{g\mathbf{E}(t_3)\cdot g\mathbf{E}(0)\}\rangle_c = \langle\{g\mathbf{E}(t_1)\cdot g\mathbf{E}(t_2)\}\,\{g\mathbf{E}(t_3)\cdot g\mathbf{E}(0)\}\rangle \\ - \frac{1}{N_c}\langle g\mathbf{E}(t_1)\cdot g\mathbf{E}(t_2)\rangle\langle g\mathbf{E}(t_3)\cdot g\mathbf{E}(0)\rangle\end{aligned}$$