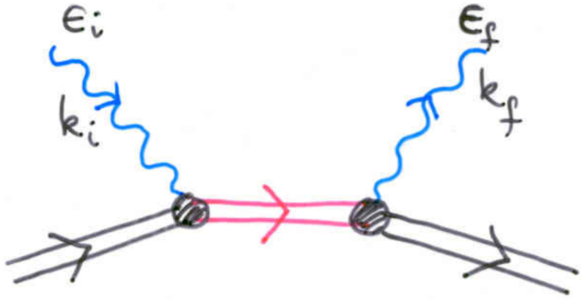
Lecture # 2

Elements of χPT

The concept of Effective Theory

Rayleigh Scattering as Example

Elastic scattering of light by atoms



(Forget e^- spin)

$$H = \frac{(\vec{P} - e\vec{A})^2}{2m_e} + e\,\Phi$$

Scattering Amplitude

Thompson term of e^-

$$W_{fi} = -\frac{e^2}{2m_e}\left\{\epsilon_f^* \cdot \epsilon_i + \frac{1}{m_e}\sum_n\left(\frac{\epsilon_f^* \cdot \langle 0|p\,e^{-ik_f x}|n\rangle\langle n|p\,e^{ik_i x}|0\rangle \cdot \epsilon_i}{\omega_i + E_0 - E_n} + \text{crossed term}\right)\right\}$$

For $\omega_i \sim \omega_f \ll E_0 - E_1$, and $|k_{i,f}| a_0 \ll 1$

use dipole approximation: $e^{kx} \to 1$

e^-'s Thomson term exactly cancelled by a term from 2^{nd} order perturbation leaving: $(\omega_i = \omega_f = \omega)$

$$W_{fi} \sim -e^2 \omega^2 \underbrace{\sum_n \left(\frac{\langle 0| x.\epsilon_f^* |n\rangle\langle n| x.\epsilon_i |0\rangle}{E_0 - E_n} + \epsilon_f^* \leftrightarrow \epsilon_i \right)}_{\text{All the Atomic details in here}}$$

Effective Hamiltonian

Requirements: Only photon field needed
Gauge invariance
Rotational invariance
Parity invariance

$$H_{effective} = -2\pi\,\alpha\,\vec{E}^2 - 2\pi\,\beta\,\vec{B}^2$$

α: electric polarizability

β: magnetic "

$$W_{fi} = -4\pi i \left(\alpha\, \omega^2 \epsilon_f^* \cdot \epsilon_i + 4\beta \left(k_f \cdot k_i\, \epsilon_f^* \cdot \epsilon_i - k_i \cdot \epsilon_f^*\, k_f \cdot \epsilon_i \right) \right)$$

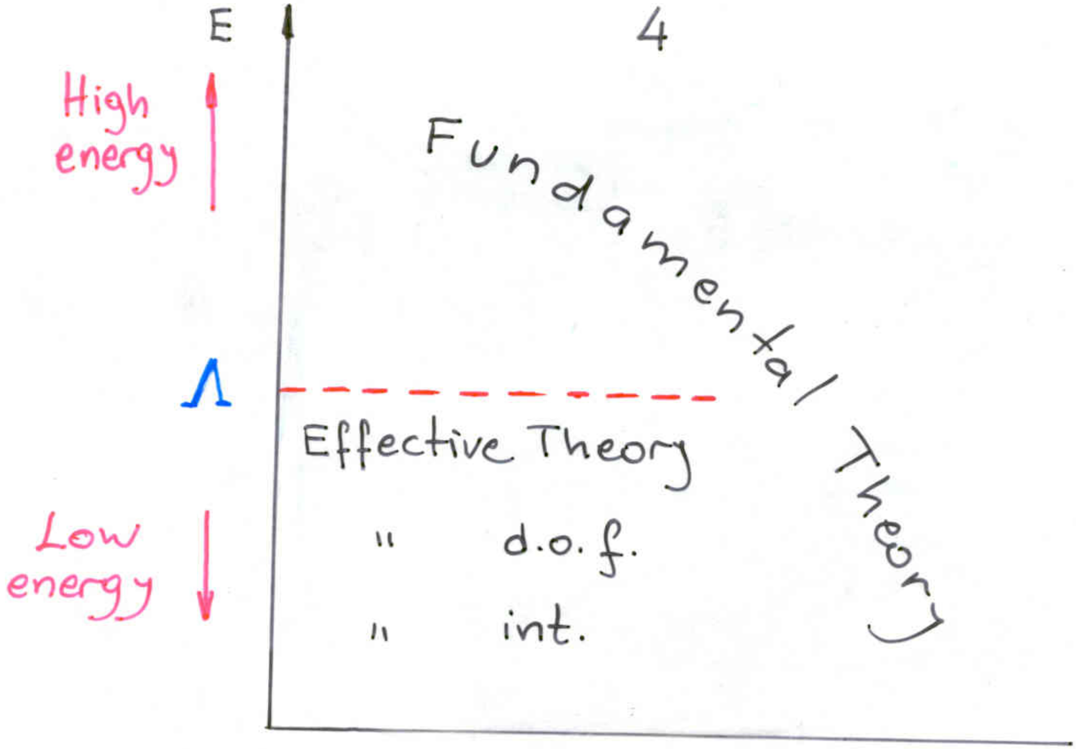
$$\sim \omega^2$$

$$\frac{d\sigma}{d\Omega} \sim \omega^4 \; *$$

Main physical features independent of details but given by general symmetry principles. Microscopic details encoded in <u>effective theory parameters</u> α & β.

Expansion in $\omega/\Delta E$ $\quad \Delta E \sim E_1 - E_0$ can be implemented with effective Hamiltonian.

* Explains why sky is blue!



Λ : matching scale. Below Λ, effective theory can be defined as expansion in powers of E/Λ. For $E < \Lambda$ effective theory is <u>fully equivalent</u> to fundamental theory.

List of requirements for Effective Theory (ETh)

1) Dynamical degrees of freedom: only those that can be excited at low energy ($E < \Lambda$).

2) All symmetries of fundamental theory must be preserved by ETh (this includes Anomalies).

3) Effective parameters or "low energy constants" (LECs) should be tuned to make ETh equivalent to fundamental theory.

Advantages of using ETh

1) It separates the high energy/small distance physics from the low energy/long distance physics.

High Energy $\longrightarrow$ LEC's
Low Energy $\longrightarrow$ dynamics of effective d.o.f.

2) Provides a hierarchical order in powers of E/Λ. This makes the physics much more transporent.

3) It is simpler to work with than full theory.

4) If full theory is not well understood at scale Λ, the determination of LEC's at that scale can be substituted by matching to experimental low energy observables.

General Scheme for Building an Effective Field Theory (EFT)

Ingredients

λ_i small parameters

G symmetry group

φ degrees of freedom at LE

$\downarrow$ Effective Lagrangian

$$\mathcal{L}_{\text{Eff.}} = \sum_n \mathcal{L}^{(n)}(\varphi, \partial\varphi, \partial^2\varphi, \dots; \lambda_i)$$

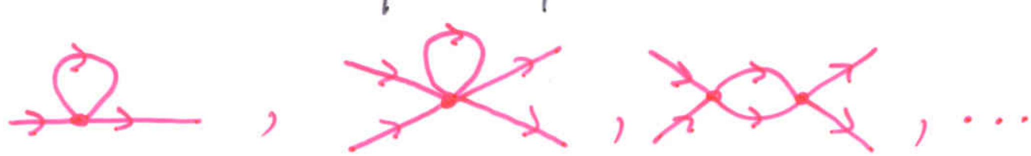
($\mathcal{L}_{\text{Eff}}$ being invariant under G)

$\mathcal{L}^{(n)}$ is $\mathcal{O}(1/\Lambda^n) \equiv \mathcal{O}(p^n)$

p: generic small $E/p/m$

$\downarrow$ QFT

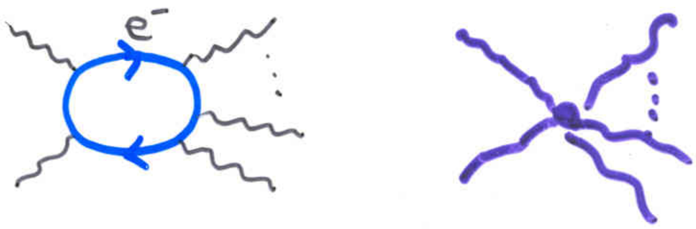
Loop expansion



Renormalization needed

Illustrative Example : Euler-Heisenberg

Low energy $\gamma\gamma \to \gamma \ldots \gamma$ processes



$E \lesssim m_e$

α, $m_e = \Lambda$ $\qquad G = U(1) \times \text{Poincaré} \times \mathcal{P} \times \mathcal{C}$

A_μ (photon field)
$(F_{\mu\nu})$

$$\mathcal{L}_{EH} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a}{\Lambda^4} \left(F_{\mu\nu} F^{\mu\nu}\right)^2$$
$$+ \frac{b}{\Lambda^4} F_{\mu\nu} F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu} + \frac{c}{\Lambda^6} F_{\mu\nu} F^{\mu\nu} \Box \left(F_{\rho\sigma} F^{\rho\sigma}\right)$$
$$+ \cdots$$

(odd # of F's forbidden by $\mathcal{C}$)

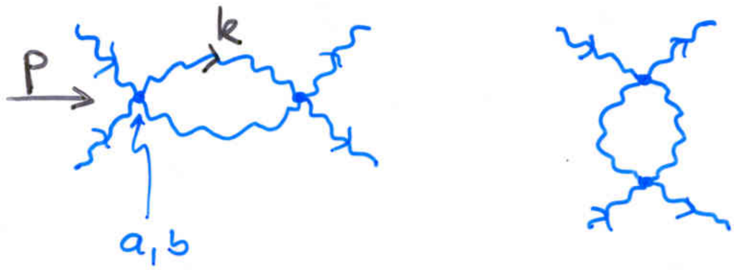
Can eliminate some higher order terms by using Eqns. of motion at lowest order

$$\partial^\mu F_{\mu\nu} = 0$$

$$\Box F_{\mu\nu} = 0$$

LECs $a, b, \ldots$ calculable from QED

Loop expansion



Loop integrals of general form:

$$\int \frac{d^4k}{(2\pi)^4} \quad \frac{k_\mu k_\nu (p-k)_\rho (p-k)_\sigma}{k^2 \; (p-k)^2}$$

UV divergent integrals (quartic div.)

UV regularization required.

Requirements on regularization

- Preserve symmetries.
- Allow for low energy power counting for loop diagrams.

EFT for low energy QCD

$$G_{QCD} = U_B(1) \times SU_L(3) \times SU_R(3) \times \text{Poincaré}$$
$$\times\ \mathcal{P} \times \mathcal{C}$$

m_u, m_d, m_s (in chiral limit all zero)

Goldstone Bosons (octet of 0^- mesons)
later → Stable Baryons (spin ½ baryon octet)

GB fields:

Manifold of QCD vacua ⟷

$$\frac{SU_L(3) \times SU_R(3)}{SU_V(3)} \sim SU(3)$$

(in Abelian Higgs model we had

$$\frac{U(1)}{\mathbb{1}} = U(1)\ )$$

$$U(x) \in SU(3)$$

$$(U^\dagger U = \mathbb{1},\ \mathrm{Tr}\, U = 0)$$

nice parametrization :

$$\Pi(x) = \pi^a(x)\, \lambda^a$$

λ^a : Gell-Mann matices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},\ \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},\ \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\mathrm{Tr}\, \lambda^a \lambda^b = 2\delta^{ab}$$

$$\Pi(x) = \begin{bmatrix} \pi_3 + \frac{\pi_8}{\sqrt{3}} & \sqrt{2}\,\pi^+ & \sqrt{2}\, K^+ \\ \sqrt{2}\,\pi^- & -\pi_3 + \frac{\pi_8}{\sqrt{3}} & \sqrt{2}\, K^0 \\ \sqrt{2}\, K^- & \sqrt{2}\, \bar{K}^0 & -\frac{2}{\sqrt{3}}\pi_8 \end{bmatrix}$$

$$U(x) = e^{i \frac{\Pi(x)}{F_0}}$$

F_0 : pion decay constant in chiral limit

Symmetry transformations of U field

$$\mathcal{P}: U(x_0, \vec{x}) = U^{\dagger}(x_0, -\vec{x})$$

$$\mathcal{C}: U(x) = U^{T}(x)$$

Chiral transf.:

$$U(x) \longrightarrow R\, U(x)\, L^{\dagger}$$

$$R \in SU_R(3) \quad , \quad L \in SU_L(3)$$

Construction of Chiral Lagrangian

First an important method : sources

Put QCD in <u>external classical</u> fields

that couple to quarks (e.g. EM field)

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + i\,\bar{q}\,\not{D}(G,l,r)\,q$$
$$-\bar{q}\,(s - i\gamma_5\, p)\,q$$

$$D_\mu(G,l,r) = \partial_\mu - ig\,G_\mu - i\,\bar{q}_L\, l_\mu\, q_L - i\,\bar{q}_R\, r_\mu\, q_R$$

sources :

$$l_\mu = l^a_\mu \frac{\lambda^a}{2} \qquad v_\mu \equiv r_\mu + l_\mu$$

$$r_\mu = r^a_\mu \frac{\lambda^a}{2} \qquad a_\mu \equiv r_\mu - l_\mu$$

$$s = s_0 + s^a \lambda^a$$
$$p = p_0 + p^a \lambda^a$$

$s \to \mathcal{M}$ when sources off.

$\mathcal{L}_{QCD}$ Locally chiral invariant

$$q_L(x) \to L(x)\, q_L(x)$$

$$q_R(x) \to R(x)\, q_R(x)$$

$$r_\mu(x) \to R(x)(r_\mu + i\partial_\mu) R^\dagger(x)$$

$$l_\mu(x) \to L(x)(l_\mu + i\partial_\mu) L^\dagger(x)$$

$$(s+ip)(x) \to R(x)\,(s+ip)(x)\, L^\dagger(x)$$

Currents :

$$L^a_\mu = \frac{\delta \mathcal{L}_{QCD}}{\delta l^{a\mu}} = \frac{1}{2}\, \bar{q}_L \gamma_\mu \lambda^a q_L$$

$$R^a_\mu = \frac{\delta \mathcal{L}_{QCD}}{\delta r^{a\mu}} = \frac{1}{2}\, \bar{q}_R \gamma_\mu \lambda^a q_R$$

$$V^a_\mu = R^a_\mu + L^a_\mu = \frac{1}{2}\, \bar{q} \gamma_\mu \lambda^a q$$

$$A^a_\mu = R^a_\mu - L^a_\mu = \frac{1}{2}\, \bar{q} \gamma_\mu \gamma_5 \lambda^a q$$

$$\bar{q} \lambda^a q = \frac{\delta \mathcal{L}_{QCD}}{\delta s^a}$$

$$\bar{q} \lambda^a \gamma_5 q = \frac{\delta \mathcal{L}}{\delta p^a}$$

Leading order chiral Lagrangian

Building blocks :

$$U$$

$$D_\mu U \equiv \partial_\mu U - i r_\mu U + i U l_\mu$$

$$\chi \equiv 2B_0 (s + ip)$$

under local $SU_L(3) \times SU_R(3)$:

$$D_\mu U \longrightarrow R D_\mu U L^\dagger$$

$$\chi \rightarrow R \chi L^\dagger$$

Lagrangian: $(\langle A \rangle \equiv \mathrm{Tr}\, A)$

$$\boxed{\mathcal{L}_\chi^{(2)} = \frac{F_0^2}{4} \left(\langle D_\mu U D^\mu U^\dagger \rangle + \langle \chi^\dagger U + \chi U^\dagger \rangle \right)}$$

used that: $UU^\dagger = \mathbb{1}$, $\langle U^\dagger D_\mu U \rangle = 0$, $\det U = 1$

Equations of Motion

$$D_\mu D^\mu U\, U^\dagger - U D_\mu D^\mu U^\dagger + U\chi^\dagger - \chi U^\dagger - \frac{1}{3}\langle U\chi^\dagger - \chi U^\dagger \rangle = 0$$

$$\Bigg\downarrow \begin{array}{l} s \to \mathcal{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \\ l, r, p \to 0 \\ 1^{st} \text{ order in } \pi \end{array}$$

$$\Box \pi + B_0 \{\mathcal{M}, \pi\} - \frac{2}{3} B_0 \langle \mathcal{M} \pi \rangle \mathbb{1} = 0$$

LO Mass Formulas:

in physical basis: $(\Box + M_{\underline{a}}^2)\, \pi_{\underline{a}} = 0$

$\pi_{\underline{a}} = \pi_a$ if $a \neq 3, 8$

$$\pi_{\underline{3}} = \pi^0 = \cos\epsilon\, \pi_3 + \sin\epsilon\, \pi_8$$
$$\pi_{\underline{8}} = \eta = -\sin\epsilon\, \pi_3 + \cos\epsilon\, \pi_8$$

$$\tan\epsilon = \frac{\sqrt{3}}{2}\, \frac{m_d - m_u}{m_s - \hat{m}}$$

$$\hat{m} = \frac{1}{2}(m_u + m_d)$$

Since $m_s \gg m_{u,d}$, ϵ is small.

$$\overset{\circ}{M}{}^2_{\pi^\pm} = 2B_0\hat{m} \quad , \quad \overset{\circ}{M}{}^2_{\pi^0} = 2B_0\hat{m} - \frac{4}{3}\frac{\sin^2\epsilon}{\cos 2\epsilon} B_0(m_s - \hat{m})$$

$$\overset{\circ}{M}{}^2_{K^\pm} = B_0(m_u + m_s) \quad , \quad \overset{\circ}{M}{}^2_{K^0} = B_0(m_d + m_s)$$

$$\overset{\circ}{M}{}^2_{\eta} = \frac{2}{3}B_0(2m_s + \hat{m}) + \frac{4}{3}\frac{\sin^2\epsilon}{\cos 2\epsilon} B_0(m_s - \hat{m})$$

$$M_{GB} \sim \sqrt{m_q}$$

Gell-Mann Okubo Relation

$$2(\overset{\circ}{M}{}^2_{K^+} + \overset{\circ}{M}{}^2_{K^0}) + 2\overset{\circ}{M}{}^2_{\pi^+} - 3(\overset{\circ}{M}{}^2_{\pi^0} + \overset{\circ}{M}{}^2_{\eta}) = 0$$

satisfied within 4%.

First important test of SXSB.

Quark mass ratios

i) $$\frac{\overset{\circ}{M}{}^2_{K^+} + \overset{\circ}{M}{}^2_{K^0}}{\overset{\circ}{M}{}^2_{\pi^+}} = \frac{m_s + \hat{m}}{2\hat{m}}$$

ii) $$\frac{\overset{\circ}{M}{}^2_{\eta} + \overset{\circ}{M}{}^2_{\pi^0}}{\overset{\circ}{M}{}^2_{\pi^+}} = \frac{4m_s + 5\hat{m}}{4\hat{m}}$$

iii) $$2\,\frac{\overset{\circ}{M}{}^2_{K^0} - \overset{\circ}{M}{}^2_{K^+}}{\overset{\circ}{M}{}^2_{\pi^+}} = \frac{m_d - m_u}{\hat{m}}$$

Electromagnetic Selfenergies

Charge Matrix: $\hat{Q} = e\begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$

Most general Lagrangian at order e^2

$$\mathcal{L}_{EM} = C \langle \hat{Q} U \hat{Q} U^{+} \rangle$$

Gives no mass shift to neutral mesons

$$\delta_{EM} M^2_{\pi^\pm} = \delta_{EM} M^2_{K^\pm} = 2\frac{C}{F_0^2} e^2$$

Dashen's Theorem

$\epsilon \to 0$

$$\delta_{EM} M_{\pi^+} = M_{\pi^+} - M_{\pi^0} = 4.59\,\text{MeV} \longrightarrow \quad C/F_0^2 \simeq 115\,\text{MeV}^2$$

$$\longrightarrow \quad \delta_{EM} M_{K^\pm} = 1.3\,\text{MeV}$$

Quark mass ratios

i) $m_s/\hat{m} = 27.7$

ii) $m_s/\hat{m} = 24.9$ $\longrightarrow$ $m_s \sim 25\,\hat{m}$; $m_u \sim 0.56\, m_d$

iii) $\dfrac{m_d - m_u}{\hat{m}} = 0.57$

What is B_0?

$$\bar{q}q = \bar{u}u + \bar{d}d + \bar{s}s = -\frac{\delta \mathcal{L}_{QCD}}{\delta s_0} = \frac{\delta \mathcal{L}_\chi^{(2)}}{\delta s_0} + \cdots$$

$$= \frac{F_0^2 B_0}{2} \langle U + U^+ \rangle$$

Order parameter:

$$\langle 0 | \bar{q}q | 0 \rangle = \frac{F_0^2 B_0}{2} \times 6$$

$$B_0 = -\frac{\langle \bar{u}u \rangle}{F_0^2} = -\frac{\langle \bar{d}d \rangle}{F_0^2} = -\frac{\langle \bar{s}s \rangle}{F_0^2}$$

$$\overset{\circ}{M}{}_\pi^2 = \hat{m} \cdot \frac{\langle \bar{u}u + \bar{d}d \rangle}{F_0^2}$$

Gell-Mann, Oakes, Renner relation

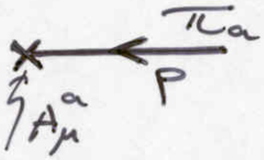
What is F_0?

$$A^a_\mu = \frac{\delta \mathcal{L}_{QCD}}{\delta a^{a\mu}} = \frac{\delta \mathcal{L}^{(2)}_\chi}{\delta a^{a\mu}} + \cdots$$

$$= -i \frac{F_0^2}{4} \langle \lambda^a (\partial_\mu U^\dagger U - \partial_\mu U U^\dagger) \rangle$$

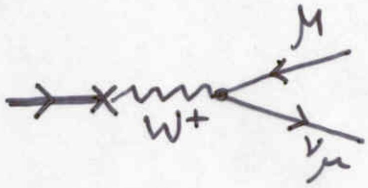
$$= - F_0 \partial_\mu \pi_a + \cdots$$

$$\langle 0 | A^a_\mu(x) | \pi_b, p \rangle = i \delta_{ab} \, p_\mu F_0 \, e^{-ip\cdot x}$$



Determination of F_π, F_K

Weak decay $\pi^+ \to \mu^+ \nu_\mu$



$$\Gamma(\pi^+ \to \mu^+ \nu_\mu) = F_\pi^2 \frac{G_F^2}{4\pi} |V_{ud}|^2 m_\mu^2 M_{\pi^+} \left(1 - \frac{m_\mu^2}{M_{\pi^+}^2}\right)\left(1 + \frac{\alpha}{\pi} C_\pi\right)$$

$$F_{\pi^+} = 92.41 \pm 0.37 \text{ MeV}$$

Similarly

$$F_{K^+} = 113.0 \pm 1.5 \text{ MeV}$$

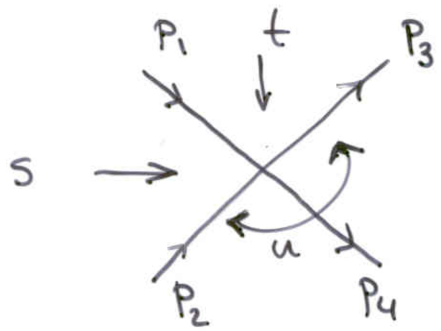
$\pi\pi$ Scattering

$$\boxed{\pi\pi \to \pi\pi}$$
$$\pi K \to \pi K$$
$$\pi\pi \to K K \quad ?$$

$$\mathcal{L}^{(2)}_{\pi\pi\to\pi\pi} = \frac{1}{48F_0^2}\left\langle [\partial_\mu \Pi, \Pi][\partial_\mu \Pi, \Pi] + 2B_0 \mathcal{M}\Pi^4 \right\rangle$$

Background:

Kinematics

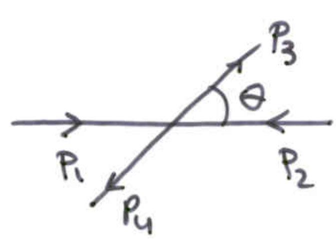


Mandelstam invariants

$$s = (p_1+p_2)^2$$
$$t = (p_1-p_3)^2$$
$$u = (p_1-p_4)^2$$

$$s+t+u = 4M_\pi^2$$

CM frame



$$p = |\vec{p}_1|$$

$$s = 4M_\pi^2 + 4p^2$$
$$t = -2p^2(1-\cos\theta)$$
$$u = -2p^2(1+\cos\theta)$$

Isospin states of $\pi\pi$: $I = 0, 1, 2$

ℓ: orbital angular momentum of $\pi\pi$ system

Bose statistics $\Longrightarrow$ $I = 0, 2$ if ℓ = even
$I = 1$ if ℓ = odd

partial wave expansion

$$T^I(s, \theta) = 32\pi \sum_\ell (2\ell+1)\, P_\ell(\cos\theta)\, t_\ell^I(s)$$

partial wave amplitudes:

$$t_\ell^I(s) = \sqrt{\frac{s}{s-4M_\pi^2}}\, \frac{1}{2i}\left(e^{2i\delta_\ell^I(s)} - 1\right)$$

effective range expansion:

$$p \operatorname{ctg} \delta = -\frac{1}{\tilde{a}} + \frac{1}{2} r_0 p^2 + \dots$$

$\tilde{a}$: scattering length

$$a \equiv -M_\pi \tilde{a}$$

Leading order results for scattering

$l = 0, 1$ only.

$$T^0(s,t,u) = \frac{1}{F_0^2}\,(2s - M_\pi^2)$$

$$T^1(s,t,u) = \frac{1}{F_0^2}\,(t-u)$$

$$T^2(s,t,u) = \frac{1}{F_0^2}\,(2M_\pi^2 - s)$$

Exercise: obtain partial wave amplitudes.

Scattering lengths to LO:

From data

$$a_0^0 = \frac{7}{32\pi}\,\frac{M_\pi^2}{F_0^2} = 0.16 \qquad 0.216 \pm 0.013$$

$$a_0^2 = -\frac{1}{16\pi}\,\frac{M_\pi^2}{F_0^2} = -0.045 \qquad -0.044 \pm 0.001$$

Second key test of SXSB.

$a_0^0 \to -0.15$ fm

Compare with NN Scattering lengths

$({}^{2S+1}L_S)$ ${}^1S_0$: $\tilde{a}_1 = -23.7$ fm $\quad {}^3S_1$: $\tilde{a}_3 = 5.42$ fm.

Observations:

- LO results give strong evidence for SXSB in QCD.
- EFT gives access to fundamental QCD parameters, namely, quark mass ratios.
- $\langle \bar{q}q \rangle$ as leading order parameter for SXSB.
- What LO lacks:
  - No π structure (point like). NLO needed for form factors
  - $F_K \neq F_\pi$ in real world.
  - Corrections to $\pi\pi$ scattering.
  - Deviations from GM-O.
  - No $\pi^0 \to \gamma\gamma$
  - etc.