The Thomas Jefferson National Accelerator Facility
Theory Group Preprint Series

Additional copies are available from the authors.

The Southeastern Universities Research Association (SURA) operates the Thomas Jefferson National Accelerator Facility for the United States Department of Energy under contract DE-AC05-84ER40150.

DISCLAIMER

This report was prepared as an account of work sponsored by the United States government. Neither the United States nor the United States Department of Energy, nor any of their employees, makes any warranty, expressed or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, mark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States government or any agency thereof.

hep-ph/9905376 17 May 1999

JLAB-THY-99-12
May 17, 1999

Evolution and Models for Skewed Parton Distributions

I.V. MUSATOV[a] and A.V. RADYUSHKIN[a,b] *
[a] *Physics Department, Old Dominion University, Norfolk, VA 23529, USA*
[b] *Jefferson Lab, Newport News, VA 23606, USA*

We discuss the structure of the "forward visible" (FW) parts of double and skewed distributions related to usual distributions through reduction relations. We use factorized models for double distributions (DDs) $\tilde{f}(x,\alpha)$ in which one factor coincides with the usual (forward) parton distribution and another specifies the profile characterizing the spread of the longitudinal momentum transfer. The model DDs are used to construct skewed parton distributions (SPDs). For small skewedness, the FW parts of SPDs $H(\bar{x},\xi)$ can be obtained by averaging forward parton densities $f(\bar{x}-\xi\alpha)$ with the weight $\rho(\alpha)$ coinciding with the profile function of the double distribution $\tilde{f}(x,\alpha)$ at small x. We show that if the x^n moments $\tilde{f}_n(\alpha)$ of DDs have the asymptotic $(1-\alpha^2)^{n+1}$ profile, then the α-profile of $\tilde{f}(x,\alpha)$ for small x is completely determined by small-x behavior of the usual parton distribution. We demonstrate that, for small ξ, the model with asymptotic profiles for $\tilde{f}_n(\alpha)$ is equivalent to that proposed recently by Shuvaev et al., in which the Gegenbauer moments of SPDs do not depend on ξ. We perform a numerical investigation of the evolution patterns of SPDs and gave interpretation of the results of these studies within the formalism of double distributions.

I. INTRODUCTION

Applications of perturbative QCD to deeply virtual Compton scattering (DVCS) and hard exclusive electroproduction processes [1–5] involve nonforward matrix elements $\langle p-r\,|\,\mathcal{O}(0,z)\,|\,p\rangle\,|_{z^2=0}$ of quark and gluon light-cone operators. They can be parametrized by two basic types of nonperturbative functions. The double distributions (DDs) $F(x,y;t)$ [2,3,6,7] specify the Sudakov light-cone "plus" fractions xp^+ and yr^+ of the initial hadron momentum p and the momentum transfer r carried by the initial parton. Treating the proportionality coefficient ζ as an independent parameter one can introduce an alternative description in terms of the nonforward parton distributions (NFPDs) $\mathcal{F}_\zeta(X;t)$ with $X = x + y\zeta$ being the total fraction of the initial hadron momentum taken by the initial parton. The shape of NFPDs explicitly depends on the parameter ζ characterizing the *skewedness* of the relevant nonforward matrix element. This parametrization of nonforward matrix elements by $\mathcal{F}_\zeta(X;t)$ is similar to that proposed by X. Ji [1] who introduced off-forward parton distributions (OFPDs) $H(\bar{x},\xi;t)$ in which the parton momenta and the skewedness parameter $\xi \equiv r^+/2P^+$ are measured in units of the average hadron momentum $P = (p+p')/2$. OFPDs and NFPDs [3,4] can be treated as particular forms of *skewed* parton distributions (SPDs). One can also introduce the version of DDs ("α-DDs" [7]) in which the active parton momentum is written in terms of symmetric variables $k = xP + (1+\alpha)r/2$.

In our approach, DDs are primary objects producing SPDs after an appropriate integration. In refs. [6,7] it was shown that using the the support and symmetry properties of DDs, one can easily establish important features of

*Also Laboratory of Theoretical Physics, JINR, Dubna, Russian Federation

SPDs such as nonanalyticity at border points $X = \zeta, 0$ [or $\tilde{x} = \pm\xi$], polynomiality of their X^N and $\tilde{x}^N$ moments in skewedness parameters ζ and ξ, etc.

The physical interpretation of DDs $F(x, y; t = 0)$ [or $f(x, \alpha; t = 0)$] and their relation to the usual parton densities $f(x)$ suggests that the x-profile of DDs $F(x, y), f(x, \alpha)$ is driven by the shape of $f(x)$ while their y or α-profile is analogous to the shape of two-body distribution amplitudes like $[y(1-x-y)]^n$, $[(1-x)^2 - \alpha^2]^n$. Fixing the profile parameter n gives simple models [6,7] for DDs which can be converted into models for SPDs.

In the present paper, our main goal is to study the self-consistency of these models with respect to the pQCD evolution. In Section II, we briefly review the basic elements of the formalism of double distributions, discuss their support and symmetry properties and relation to usual parton densities. In Section III, we describe factorized profile models for double distributions and give explicit model expressions for skewed distributions. In Section IV we consider a practically important case when skewedness parameters ζ or ξ are small. The factorized models for DDs in this case can be taken in a very simple form $\tilde{f}(x, \alpha; t = 0) = f(x)\rho(\alpha)$, where $\rho(\alpha)$ is a normalized profile function. As a result, SPDs $\tilde{\mathcal{F}}_\zeta(X)$ (or $H(\tilde{x}, \xi)$) in this model are obtained by averaging the relevant forward distribution $\tilde{f}(x)$ over the interval $(X - \zeta, X)$ [or $(\tilde{x} - \xi, \tilde{x} + \xi)$] with the weight $\rho(\alpha)$ (we use the convention [7] that "tilded" parton distributions are those defined on the $(-1, 1)$ interval). In Section V, we study the impact of the pQCD evolution on the profile function $\rho(\alpha)$. Since the α-DDs are hybrids which look like usual parton densities wrt x and like distribution amplitudes wrt α, the simplest renormalization properties at one loop have the combined $x^m C_l^{3/2+m}(\alpha)$ moments of $\tilde{f}(x, \alpha)$. As a result, independently of the initial condition, the α-profile of the x^n moment $\tilde{f}_n(\alpha)$ of the α-DD $\tilde{f}(x, \alpha)$ under the pQCD evolution asymptotically tends to $(1 - \alpha^2)^{n+1}$. We investigate the "asymptotic profile model" in which $\tilde{f}_n(\alpha)$ are given by their asymptotic form and show that it imposes a remarkable correlation $\tilde{f}(x, \alpha) = F(x/(1 - \alpha^2))$ between the x-dependence of the α-DDs and their α-profile. To study the impact of pQCD evolution on the DD based models of SPDs, we perform an explicit numerical evolution of SPDs. In Section VI, we describe a simple algorithm for the leading-log evolution of SPDs based on direct iterative convolutions of evolution kernels $W_\zeta(X, Z)$ with SPDs $\mathcal{F}_\zeta(Z)$. In section VII, we discuss the results of our numerical calculations. In Appendix A, we show that the approximation (used in Ref. [8]) in which the Gegenbauer moments of SPDs do not depend on skewedness, is equivalent to the asymptotic profile model for DDs. In Appendix B, we present explicit form of evolution equations for SPDs used in our numerical calculations.

II. DOUBLE DISTRIBUTIONS

In the pQCD factorization treatment of hard electroproduction processes, the nonperturbative information is accumulated in the nonforward matrix elements $\langle p - r \,|\, \mathcal{O}(0, z) \,|\, p \rangle$ of light cone operators $\mathcal{O}(0, z)$. For $z^2 = 0$ the matrix elements depend on the relative coordinate z through two Lorentz invariant variables (pz) and (rz). In the forward case, when $r = 0$, one obtains the usual quark helicity-averaged densities by Fourier transforming the relevant matrix element with respect to (pz)

$$\langle p, s' \mid \bar{\psi}_a(0)\hat{z}E(0, z; A)\psi_a(z) \mid p, s\rangle \,|_{z^2=0} = \bar{u}(p, s')\hat{z}u(p, s) \int_0^1 \left(e^{-iz(pz)} f_a(x) - e^{iz(pz)} f_{\bar{a}}(x)\right) dx \,, \tag{2.1}$$

where $E(0, z; A)$ is the gauge link, $\bar{u}(p', s'), u(p, s)$ are the Dirac spinors and we use the notation $\gamma_\alpha z^\alpha \equiv \hat{z}$. The functions $f_a(x)$, $f_{\bar{a}}(x)$ can be also treated as components of the "tilded" distribution $\tilde{f}_a(x) = f_a(x)\theta(x > 0) - f_{\bar{a}}(x)\theta(x < 0)$, whose support extends to $[-1 \le x \le 1]$. In the nonforward case, we can use the double Fourier representation with respect to both (pz) and (rz) [3]:

$$\begin{aligned} &\langle p', s' \mid \bar{\psi}_a(0)\hat{z}E(0, z; A)\psi_a(z) \mid p, s\rangle \,|_{z^2=0} \\ &= \bar{u}(p', s')\hat{z}u(p, s) \int_0^1 dy \int_{-1}^1 e^{-iz(pz)-iy(rz)}\, \tilde{F}_a(x, y; t)\, \theta(0 \le x + y \le 1)\, dx \;+\; \text{“}O(r)\text{−terms”} \,, \end{aligned} \tag{2.2}$$

where the "$O(r)$-terms" stands for contributions which have the structure $\bar{u}(p', s')r^\mu z^\nu \sigma_{\mu\nu} u(p, s)\Phi((pz), (rz))$ [1] and $(rz)\bar{u}(p', s')u(p, s)\Psi((rz))$ [13,9] and vanish in the $r \to 0$ limit.

For any Feynman diagram, the spectral constraints $-1 \le x \le 1$, $0 \le y \le 1$, $0 \le x + y \le 1$ were proved in the α-representation [3] using the approach of Ref. [10]. The support area for the *double distribution* $\tilde{F}_a(x, y; t)$ is shown on Fig.1*a*.

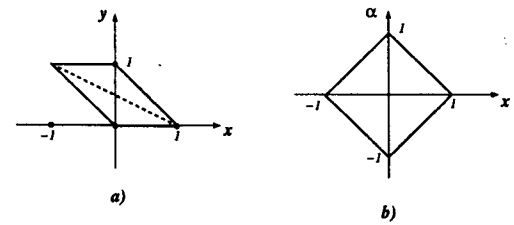


FIG. 1. *a*) Support region and symmetry line $y = x/2$ for y-DDs $\tilde{F}(x, y; t)$. *b*) Support region for α-DDs $\tilde{f}(x, \alpha)$.

Taking the $r = 0$ limit of Eq. (2.2), one obtains "reduction formulas" relating the double distribution $\tilde{F}_a(x, y; t = 0)$ to the quark and antiquark parton densities

$$\int_0^{1-x} \tilde{F}_a(x, y; t = 0)|_{x>0}\, dy = f_a(x) \qquad ; \qquad \int_{-x}^1 \tilde{F}_a(x, y; t = 0)|_{x<0}\, dy = -f_{\bar{a}}(-x) \,. \tag{2.3}$$

Hence, the positive-x and negative-x components of the double distribution $\tilde{F}_a(x, y; t)$ can be treated as nonforward generalizations of quark and antiquark densities, respectively. If we define the "untilded" DDs

$$F_a(x, y; t) = \tilde{F}_a(x, y; t)|_{x>0} \qquad ; \qquad F_{\bar{a}}(x, y; t) = -\tilde{F}_a(-x, 1 - y; t)|_{x<0} \,, \tag{2.4}$$

then x is always positive and the reduction formulas have the same form

$$\int_0^{1-x} F_{a,\bar{a}}(x, y; t = 0)|_{x\neq 0}\, dy = f_{a,\bar{a}}(x) \tag{2.5}$$

in both cases. The new distributions both "live" on the triangle $0 \le x, y \le 1$, $0 \le x + y \le 1$. Taking z in the lightcone "minus" direction, we arrive at the parton interpretation of functions $F_{a,\bar{a}}(x, y; t)$ as probability amplitudes for an outgoing parton to carry the fractions xp^+ and yr^+ of the external momenta r and p. The double distributions $F(x, y; t)$ are universal functions describing the flux of p^+ and r^+ independently of the ratio r^+/p^+.

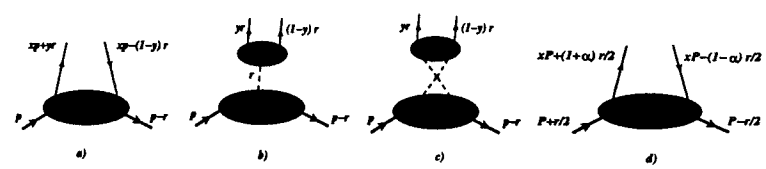


FIG. 2. *a*) Parton picture in terms of y-DDs; *b*) meson-like contribution; *c*) Polyakov-Weiss contribution; *d*) parton picture in terms of α-DDs.

The functions $\tilde{F}(x, y; t)$ may have singular terms at $x = 0$ proportional to $\delta(x)$ or its derivative(s). In such terms the sensitivity of the parton distribution to the plus-component of the initial hadron momentum is lost, and they

have no projection onto the usual parton densities. We will denote them by $F_M(x,y;t)$ – they may be interpreted as coming from the t-channel meson-exchange type contributions (see Fig.2b). As shown by Polyakov and Weiss [9], the terms with quartic pion vertex (Fig.2c), in which the dependence on (pz) is also lost, correspond to an independent $(rz)\bar{u}(p',s')u(p,s)\Psi((rz))$ type contribution. Both the meson-exchange and Polyakov-Weiss terms are invisible in the forward limit, hence the existing knowledge of the usual parton densities cannot be used to constrain these terms. In what follows, we consider only the "forward visible parts" of SPDs which are obtained by scanning the $x \neq 0$ parts of the relevant DDs.

To make the description more symmetric with respect to the initial and final hadron momenta, we can treat nonforward matrix elements as functions of (Pz) and (rz), where $P = (p+p')/2$ is the average hadron momentum. The relevant double distributions $\tilde{f}_a(x,\alpha;t)$ [which we will call α-DDs to distinguish them from y-DDs $F(x,y;t)$] are defined by [11,7]

$$\langle p'|\bar{\psi}_a(-z/2)\hat{z}\psi_a(z/2)|p\rangle = \bar{u}(p')\hat{z}u(p)\int_{-1}^{1}dx\int_{-1+|x|}^{1-|x|}e^{-ix(Pz)-i\alpha(rz)/2}\tilde{f}_a(x,\alpha;t)\,d\alpha + \text{"}O(r)\text{–terms"}. \tag{2.6}$$

The support area for $\tilde{f}_a(x,\alpha;t)$ is shown in Fig.1b. Again, the usual forward densities $f_a(x)$ and $f_{\bar{a}}(x)$ are given by integrating $\tilde{f}_a(x,\alpha;t=0)$ over vertical lines $x=$ const for $x>0$ and $x<0$, respectively. Due to hermiticity and time-reversal invariance properties of nonforward matrix elements, the α-DDs are even functions of α:

$$\tilde{f}_a(x,\alpha;t) = \tilde{f}_a(x,-\alpha;t).$$

For our original y-DDs $F_{a,\bar{a}}(x,y;t)$, this corresponds to the "Munich" symmetry with respect to the interchange $y \leftrightarrow 1-x-y$ established in Ref. [12]. The a-quark contribution into the flavor-singlet operator can be parametrized either by y-DDs $\tilde{F}_a^S(x,y;t)$ or by α-DDs $\tilde{f}_a^S(x,\alpha;t)$. The latter are even functions of α and odd functions of x:

$$\tilde{f}_a^S(x,\alpha;t) = \{f_a(|x|,|\alpha|;t) + f_{\bar{a}}(|x|,|\alpha|;t)\}\,\mathrm{sign}(x) + f_M^S(x,\alpha;t). \tag{2.7}$$

The valence quark functions $\tilde{f}_a^V(x,\alpha;t)$ are even functions of both α and x:

$$\tilde{f}_a^V(x,\alpha;t) = f_a(|x|,|\alpha|;t) - f_{\bar{a}}(|x|,|\alpha|;t) + f_M^V(x,\alpha;t). \tag{2.8}$$

It is convenient to define the gluonic α-DD $\tilde{f}^G(x,\alpha;t)$ in such a way that its integral over α for $t=0$, also gives the usual forward gluon density $\tilde{f}^G(x)$:

$$\begin{aligned}&\langle P-r/2\,|\,z_\mu z_\nu G^a_{\mu\alpha}(-z/2)E_{ab}(-z/2,z/2;A)G^b_{\alpha\nu}(z/2)\,|\,P+r/2\rangle\,|_{z^2=0} \\ &= \bar{u}(p')\hat{z}u(p)\,(Pz)\int_0^1 dx\int_{-1+|x|}^{1-|x|}e^{-ix(Pz)-i\alpha(rz)/2}\,x\tilde{f}^G(x,\alpha;t)\,d\alpha + O(r) - \text{terms}.\end{aligned} \tag{2.9}$$

The gluon SPD $H^G(\tilde{x},\xi;\mu)$ is constructed in this case from $x\tilde{f}^G(x,\alpha;t)$. Just like the singlet quark distribution, the function $\tilde{f}^G(x,\alpha;t)$ is an odd function of x.

III. MODELS FOR DOUBLE AND SKEWED DISTRIBUTIONS

The reduction formulas and interpretation of the x-variable as the fraction of p (or P) momentum suggest that the profile of $F(x,y)$ (or $f(x,\alpha)$) in x-direction is basically determined by the shape of $f(x)$. On the other hand, the profile in y (or α) direction characterizes the spread of the parton momentum induced by the momentum transfer r. In particular, since the α-DDs $\tilde{f}(x,\alpha)$ are even functions of α, it make sense to write

$$\tilde{f}(x,\alpha) = h(x,\alpha)\,\tilde{f}(x), \tag{3.1}$$

where $h(x,\alpha)$ is an even function of α normalized by

$$\int_{-1+|x|}^{1-|x|} h(x,\alpha)\,d\alpha = 1. \tag{3.2}$$

We may expect that the α-profile of $h(x,\alpha)$ is similar to that of a symmetric distribution amplitude (DA) $\varphi(\alpha)$. Since $|\alpha| \le 1-|x|$, to get a more complete analogy with DA's, it makes sense to rescale α as $\alpha = (1-|x|)\beta$ introducing the variable β with x-independent limits: $-1 \le \beta \le 1$. The simplest model is to assume that the profile in the β-direction is a universal function $g(\beta)$ for all x. Possible simple choices for $g(\beta)$ may be $\delta(\beta)$ (no spread in β-direction), $\frac{3}{4}(1-\beta^2)$ (characteristic shape for asymptotic limit of nonsinglet quark distribution amplitudes), $\frac{15}{16}(1-\beta^2)^2$ (asymptotic shape of gluon distribution amplitudes), etc. In the variables x,α, this gives

$$h^{(\infty)}(x,\alpha) = \delta(\alpha)\ ,\ h^{(1)}(x,\alpha) = \frac{3}{4}\frac{(1-|x|)^2-\alpha^2}{(1-|x|)^3}\ ,\ h^{(2)}(x,\alpha) = \frac{15}{16}\frac{[(1-|x|)^2-\alpha^2]^2}{(1-|x|)^5}\ . \tag{3.3}$$

These models can be treated as specific cases of the general profile function

$$h^{(b)}(x,\alpha) = \frac{\Gamma(2b+2)}{2^{2b+1}\Gamma^2(b+1)}\frac{[(1-|x|)^2-\alpha^2]^b}{(1-|x|)^{2b+1}}, \tag{3.4}$$

whose width is governed by the parameter b.

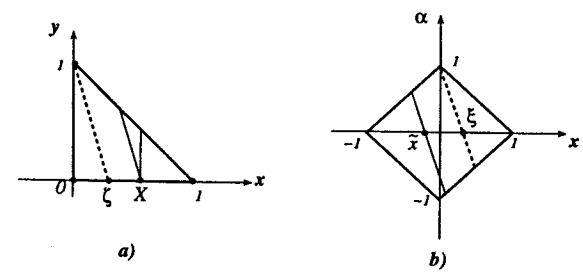


FIG. 3. Integration lines for integrals relating SPDs and DDs.

The coefficient of proportionality $\zeta = r^+/p^+$ (or $\xi = r^+/2P^+$) between the plus components of the momentum transfer and initial (or average) momentum specifies the *skewedness* of the matrix elements. The characteristic feature implied by representations for double distributions [see, e.g., Eq.(2.2)] is the absence of the ζ-dependence in the DDs $F(x,y)$ and ξ-dependence in $f(x,\alpha)$. An alternative way to parametrize nonforward matrix elements of light-cone operators is to use ζ (or ξ) and the *total* momentum fractions $X \equiv x+y\zeta$ (or $\tilde{x} \equiv x+\xi\alpha$) as independent variables. Integrating each particular double distribution over y gives the nonforward parton distributions

$$\begin{aligned}\mathcal{F}_\zeta^i(X) &= \int_0^1 dx\int_0^{1-x}\delta(x+\zeta y-X)\,F_i(x,y)\,dy \\ &= \theta(X\ge\zeta)\int_0^{X/\bar{\zeta}}F_i(X-y\zeta,y)\,dy + \theta(X\le\zeta)\int_0^{X/\zeta}F_i(X-y\zeta,y)\,dy,\end{aligned} \tag{3.5}$$

where $\bar{\zeta} \equiv 1-\zeta$. The two components of NFPDs correspond to positive $(X>\zeta)$ and negative $(X<\zeta)$ values of the fraction $X' \equiv X-\zeta$ associated with the "returning" parton. As explained in refs. [2,3], the second component can be interpreted as the probability amplitude for the initial hadron with momentum p to split into the final

hadron with momentum $(1-\zeta)p$ and the two-parton state with total momentum $r = \zeta p$ shared by the partons in fractions Yr and $(1-Y)r$, where $Y = X/\zeta$.

The relation between "untilded" NFPDs and DDs can be illustrated on the "DD-life triangle" defined by $0 \le x, y, x+y \le 1$ (see Fig.3a). Specifically, to get $\mathcal{F}_\zeta(X)$, one should integrate $F(x,y)$ over y along a straight line $x = X - \zeta y$. Fixing some value of ζ, one deals with a set of parallel lines intersecting the x-axis at $x = X$. The upper limit of the y-integration is determined by intersection of this line either with the line $x+y=1$ (this happens if $X > \zeta$) or with the y-axis (if $X < \zeta$). The line corresponding to $X = \zeta$ separates the triangle into two parts generating the two components of the nonforward parton distribution.

In a similar way, we can write the relation between OFPDs $H(\tilde{x},\xi;t)$ and the α-DDs $\tilde{f}(x,\alpha;t)$

$$H(\tilde{x},\xi;t) = \int_{-1}^{1} dx \int_{-1+|x|}^{1-|x|} \delta(x + \xi\alpha - \tilde{x})\, \tilde{f}(x,\alpha;t)\, d\alpha\,. \tag{3.6}$$

It should noted that OFPDs as defined by X. Ji [1] correspond to parametrization of the nonforward matrix element by a Fourier integral with a single common exponential with $-1 \le \tilde{x} \le 1$, i.e., OFPDs are equivalent to "tilded" NFPDs. The delta-function in Eq.(3.6) specifies the line of integration in the $\{x,\alpha\}$ plane. For definiteness, we will assume below that ξ is positive.

Information contained in SPDs originates from two physically different sources: meson-exchange type contributions $\mathcal{F}_\zeta^M(X)$ coming from the singular $x=0$ parts of DDs and the functions $\mathcal{F}_\zeta^a(X)$, $\mathcal{F}_\zeta^{\bar{a}}(X)$ obtained by scanning the $x \ne 0$ parts of DDs $F^a(x,y)$, $F^{\bar{a}}(x,y)$. The support of exchange contributions is restricted to $0 \le X \le \zeta$. Up to rescaling, the function $\mathcal{F}_\zeta^M(X)$ has the same shape for all ζ. For any nonvanishing X, these exchange terms become invisible in the forward limit $\zeta \to 0$. On the other hand, the support of functions $\mathcal{F}_\zeta^a(X)$, $\mathcal{F}_\zeta^{\bar{a}}(X)$ in general covers the whole $0 \le X \le 1$ region. Furthermore, the forward limit of such SPDs as $\mathcal{F}_\zeta^{a,\bar{a}}(X)$ is rather well known from inclusive measurements. Hence, information contained in the usual (forward) densities $f^a(x)$, $f^{\bar{a}}(x)$ can be used to restrict the models for $\mathcal{F}_\zeta^a(X)$, $\mathcal{F}_\zeta^{\bar{a}}(X)$.

Let us consider SPDs constructed using simple models of DDs specified in Section III. In particular, the model $f^{(\infty)}(x,\alpha) = \delta(\alpha) f(x)$ (equivalent to $F^{(\infty)}(x,y) = \delta(y - \tilde{x}/2) f(x)$), gives the simplest model $H^{(\infty)}(\tilde{x},\xi;t=0) = f(x)$ in which OFPDs at $t=0$ have no ξ-dependence. For NFPDs this gives

$$\mathcal{F}_\zeta^{(\infty)}(X) = \frac{\theta(X \ge \zeta/2)}{1-\zeta/2} f\left(\frac{X-\zeta/2}{1-\zeta/2}\right), \tag{3.7}$$

i.e., NFPDs for non-zero ζ are obtained from the forward distribution $f(X) \equiv \mathcal{F}_{\zeta=0}(X)$ by shift and rescaling.

In case of the $b=1$ and $b=2$ models, simple analytic results can be obtained only for some explicit forms of $f(x)$. For the "valence quark"-oriented ansatz $\tilde{f}^{(1)}(x,\alpha)$, the following choice of a normalized distribution

$$f^{(1)}(x) = \frac{\Gamma(5-a)}{6\,\Gamma(1-a)}\, x^{-a}(1-x)^3 \tag{3.8}$$

is both close to phenomenological quark distributions and produces a simple expression for the double distribution since the denominator $(1-x)^3$ factor in Eq. (3.3) is canceled. As a result, the integral in Eq. (3.6) is easily performed and we get

$$H_V^1(\tilde{x},\xi)|_{|\tilde{x}|\ge\xi} = \frac{1}{\xi^3}\left(1-\frac{a}{4}\right)\left\{\left[(2-a)\xi(1-\tilde{x})(x_1^{2-a}+x_2^{2-a}) + (\xi^2-\tilde{x})(x_1^{2-a}-x_2^{2-a})\right]\theta(\tilde{x}) + (\tilde{x}\to-\tilde{x})\right\} \tag{3.9}$$

for $|\tilde{x}| \ge \xi$ and

$$H_V^1(\tilde{x},\xi)|_{|\tilde{x}|\le\xi} = \frac{1}{\xi^3}\left(1-\frac{a}{4}\right)\left\{x_1^{2-a}[(2-a)\xi(1-\tilde{x}) + (\xi^2-x)] + (\tilde{x}\to-\tilde{x})\right\} \tag{3.10}$$

in the middle $-\xi \le \tilde{x} \le \xi$ region. We use here the notation $x_1 = (\tilde{x}+\xi)/(1+\xi)$ and $x_2 = (\tilde{x}-\xi)/(1-\xi)$ [13]. To extend these expressions onto negative values of ξ, one should substitute ξ by $|\xi|$. One can check, however, that no odd powers of $|\xi|$ would appear in the $\tilde{x}^N$ moments of $H^{1V}(\tilde{x},\xi)$. Furthermore, these expressions are explicitly

non-analytic for $x = \pm\xi$. This is true even if a is integer. Discontinuity at $x = \pm\xi$, however, appears only in the second derivative of $H^{1V}(\tilde{x},\xi)$, i.e., the model curves for $H^{1V}(\tilde{x},\xi)$ look very smooth (see Fig.4). The explicit expressions for NFPDs in this model were given in ref. [6]. The relevant curves are also shown in Fig.4.

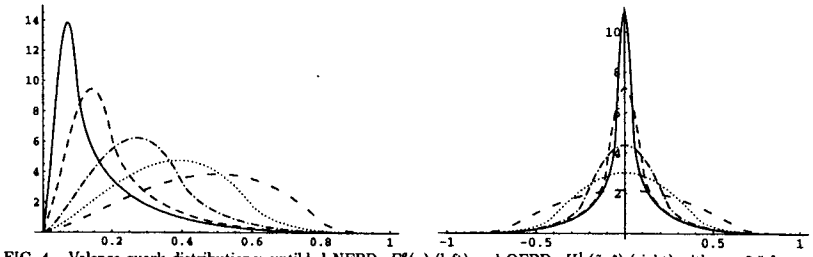


FIG. 4. Valence quark distributions: untilded NFPDs $F_\zeta^a(x)$ (left) and OFPDs $H_V^1(\tilde{x},\xi)$ (right) with $a=0.5$ for several values of ζ, namely, 0.1, 0.2, 0.4, 0.6, 0.8 and corresponding values of $\xi = \zeta/(2-\zeta)$. Lower curves correspond to larger values of ζ.

For $a=0$, the $x > \xi$ part of OFPD has the same x-dependence as its forward limit, differing from it by an overall ξ-dependent factor only:

$$H^{1V}(\tilde{x},\xi)|_{a=0} = 4\,\frac{(1-|\tilde{x}|)^3}{(1-\xi^2)^2}\,\theta(|\tilde{x}|\ge\xi) + 2\,\frac{\xi+2-3\tilde{x}^2/\xi}{(1+\xi)^2}\,\theta(|\tilde{x}|\le\xi)\,. \tag{3.11}$$

The $(1-|\tilde{x}|)^3$ behaviour can be trivially continued into the $|\tilde{x}| < \xi$ region. However, the actual behaviour of $H^{1V}(\tilde{x},\xi)|_{a=0}$ in this region is given by a different function. In other words, $H^{1V}(\tilde{x},\xi)|_{a=0}$ can be represented as a sum of a function analytic at border points and a contribution whose support is restricted by $|\tilde{x}| \le \xi$. It should be emphasized that despite its DA-like appearance, this contribution should not be treated as an exchange-type term. It is generated by regular $x \ne 0$ part of DD, and, unlike $\varphi(\tilde{x}/\xi)/\xi$ functions changes its shape with ξ becoming very small for small ξ.

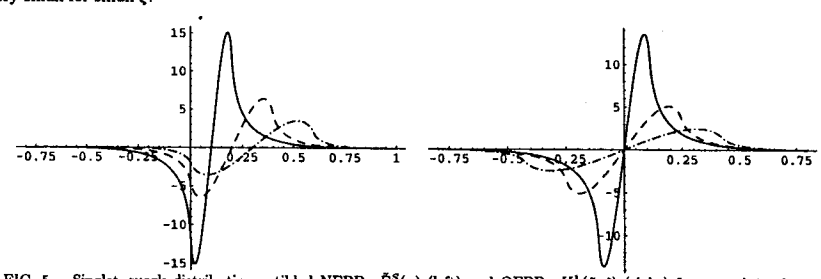


FIG. 5. Singlet quark distributions: tilded NFPDs $\tilde{F}_\zeta^S(x)$ (left) and OFPDs $H_S^1(\tilde{x},\xi)$ (right) for several ζ values 0.2, 0.4, 0.6 and corresponding values of $\xi = \zeta/(2-\zeta)$. Lower curves correspond to larger values of ζ. Forward distribution is modeled by $(1-x)^3/x$.

For the singlet quark distribution, the α-DDs $\tilde{f}^S(x,\alpha)$ should be odd functions of x. Still, we can use the model like (3.8) for the $x > 0$ part, but take $\tilde{f}^S(x,\alpha)|_{x\ne0} = A\, f^{(1)}(|x|,\alpha)\,\mathrm{sign}(x)$. Note, that the integral (3.6) producing $H^S(\tilde{x},\xi)$ in the $|\tilde{x}| \le \xi$ region would diverge for $\alpha \to \tilde{x}/\xi$ if $a \ge 1$, which is the usual case for standard parametrizations of singlet quark distributions for sufficiently large Q^2. However, due to the antisymmetry of

$\tilde{f}^S(x,\alpha)$ wrt $x \to -x$ and its symmetry wrt $\alpha \to -\alpha$, the singularity at $\alpha = \tilde{x}/\xi$ can be integrated using the principal value prescription which in this case produces the $x \to -x$ antisymmetric version of Eqs.(3.9) and (3.10). For $a = 0$, its middle part reduces to

$$H^{1S}(\tilde{x},\xi)|_{|\tilde{x}|\le\xi,a=0} = 2x\,\frac{3\xi^2 - 2x^2\xi - x^2}{\xi^3(1+\xi)^2}\,. \tag{3.12}$$

The shape of singlet SPDs in this model is shown in Fig. 5

IV. SPDS AT SMALL SKEWEDNESS

To study the deviation of skewed distributions from their forward counterparts for small ξ (or ζ), let us consider the $x \ge \xi$ part of $H(x,\xi)$ [see Eq.(3.6)] and use its expansion in powers of ξ [6]:

$$H(\tilde{x};\xi)|_{\tilde{x}\ge\xi} = f(\tilde{x}) + \xi^2\left[\frac{1}{2}\int_{-(1-\tilde{x})}^{(1-\tilde{x})}\frac{\partial^2 f(\tilde{x},\alpha)}{\partial\tilde{x}^2}\,\alpha^2\,d\alpha + (1-\tilde{x})^2\left(\frac{\partial f(\tilde{x},\alpha)}{\partial\alpha} - 2\frac{\partial f(\tilde{x},\alpha)}{\partial\tilde{x}}\right)\Big|_{\alpha=1-\tilde{x}}\right] + \dots\,. \tag{4.1}$$

where $f(\tilde{x})$ is the forward distribution. For small ξ, the corrections are formally $O(\xi^2)$. However, if $f(x,\alpha)$ has a singular behavior like x^{-a}, then

$$\frac{\partial^2 f(\tilde{x},\alpha)}{\partial\tilde{x}^2} \sim \frac{a(1+a)}{\tilde{x}^2}f(\tilde{x},\alpha)\,,$$

and the relative suppression of the first correction is $O(\xi^2/\tilde{x}^2)$. Though the corrections are tiny for $\tilde{x} \gg \xi$, in the region $\tilde{x} \sim \xi$ it has no parametric smallness. It is easy to write expicitly all the terms which are not suppressed in the $\tilde{x} \sim \xi \to 0$ limit

$$H(\tilde{x};\xi) = \sum_{k=0}\frac{\xi^{2k}}{(2k)!}\int_{-1}^{1}\frac{\partial^{2k}f(\tilde{x},\alpha)}{\partial\tilde{x}^{2k}}\,\alpha^{2k}\,d\alpha + \dots = \int_{-1}^{1}\tilde{f}(\tilde{x}-\xi\alpha,\alpha)\,d\alpha + \dots\,, \tag{4.2}$$

where the ellipses denote the terms vanishing in this limit. This result can be directly obtained from Eq. (3.6) by noting that for small x, we can neglect the x-dependence in the limits $\pm(1-|x|)$ of the α-integration. Furthermore, for small x one can also neglect the x-dependence of the profile function $h(x,\alpha)$ in Eq. (3.1) and take the model $\tilde{f}(x,\alpha) = \tilde{f}(x)\rho(\alpha)$ with $\rho(\alpha)$ being a symmetric normalized weight function on $-1 \le \alpha \le 1$. Hence, in the region where both $\tilde{x}$ and ξ are small, we can approximate Eq. (3.6) by

$$H(\tilde{x};\xi) = \text{"P"}\int_{-1}^{1}\tilde{f}(\tilde{x}-\xi\alpha)\rho(\alpha)\,d\alpha + \dots\,, \tag{4.3}$$

i.e., the OFPD $H(\tilde{x};\xi)$ is obtained in this case by averaging the usual (forward) parton density $f(x)$ over the region $\tilde{x}-\xi \le x \le \tilde{x}+\xi$ with the weight $\rho(\alpha)$. The principal value prescription "P" is only necessary in the case of singular quark singlet distributions which are odd in x. In terms of NFPDs, the relation is

$$\tilde{\mathcal{F}}_\zeta(X) = \text{"P"}\int_{-1}^{1}\tilde{f}(X-\zeta(1+\alpha)/2)\rho(\alpha)\,d\alpha + \dots\,, \tag{4.4}$$

i.e., the average is taken over the region $X-\zeta \le x \le X$.

In fact, for small values of the skewedness parameters ξ, ζ, Eqs. (4.3), (4.4) can be used for all values of $\tilde{x}$ and X: if $\tilde{x} \gg \xi$, Eq. (4.3) gives the correct result $H(\tilde{x};\xi) = \tilde{f}(\tilde{x}) + O(\xi^2)$. Hence, to get the "forward visible" parts of SPDs at small skewedness, one only needs to know the shape of the normalized profile function $\rho(\alpha)$.

The imaginary part of hard exclusive meson electroproduction amplitude is determined by the skewed distributions at the border point. For this reason, the magnitude of $\mathcal{F}_\zeta(\zeta)$ [or $H(\xi,\xi)$] and its relation to the forward densities $f(x)$ has a practical interest. This example also gives a possibility to study the sensititivity of the results to the

choice of the profile function. Assuming the infinitely narrow weight $\rho(\alpha) = \delta(\alpha)$, we have $\mathcal{F}_\zeta(X) = f(X-\zeta/2)+\dots$ and $H(x,\xi) = f(x)$. Hence, both $\mathcal{F}_\zeta(\zeta)$ and $H(\xi,\xi)$ are given by $f(x_{Bj}/2)$ since $\zeta = x_{Bj}$ and $\xi = x_{Bj}/2+\dots$. Since the argument of $f(x)$ is twice smaller than in deep inelastic scattering, this results in an enhancement factor. In particular, if $f(x) \sim x^{-a}$ for small x, the ratio $\mathcal{R}(\zeta) \equiv \mathcal{F}_\zeta(\zeta)/f(\zeta)$ is 2^a. The use of a wider profile function $\rho(\alpha)$ produces further enhancement. For example, taking the normalized profile function

$$\rho_b(\alpha) \equiv \frac{\Gamma(b+3/2)}{\Gamma(1/2)\Gamma(b+1)}(1-\alpha^2)^b = \frac{\Gamma(2b+2)}{2^{2b+1}\Gamma^2(b+1)}(1-\alpha^2)^b \tag{4.5}$$

and $f(x) \sim x^{-a}$ we get

$$\mathcal{R}^{(b)}(\zeta) \equiv \frac{\mathcal{F}_\zeta^{(b)}(\zeta)}{f(\zeta)} = \frac{\Gamma(2b+2)\Gamma(b-a+1)}{\Gamma(2b-a+2)\Gamma(b+1)} \tag{4.6}$$

which is larger than 2^a for any finite b and $0 < a < 2$. The 2^a enhancement appears as the $n \to \infty$ limit of Eq.(4.5). For small integer n, Eq.(4.5) reduces to simple formulas obtained in refs. [6,7]. For $n = 1$, we have

$$\frac{\mathcal{F}_\zeta^{(b=1)}(\zeta)}{f(\zeta)} = \frac{1}{(1-a/2)(1-a/3)}\,, \tag{4.7}$$

which gives the factor of 3 for the enhancement if $a = 1$. For $b = 2$, the ratio (4.5) becomes

$$\frac{\mathcal{F}_\zeta^{(b=2)}(\zeta)}{f(\zeta)} = \frac{1}{(1-a/3)(1-a/4)(1-a/5)}\,, \tag{4.8}$$

producing a smaller enhancement factor 5/2 for $a = 1$. Calculating the enhancement factors, one should remember that the gluon SPD $\mathcal{F}_\zeta(X)$ reduces to $Xf_g(X)$ in the $\zeta = 0$ limit. Hence, to get the enhancement factor corresponding to the $f_g(x) \sim x^{-\lambda}$ small-x behavior of the forward gluon density, one should take $a = \lambda - 1$ in Eq.(4.5), i.e., despite the fact that the $1/x$ behavior of the singlet quark distribution gives the factor of 3 for the $\mathcal{R}^{(1)}(\zeta)$ ratio, the same shape of the gluon distribution results in no enhancement.

Due to evolution, the effective parameter a characterizing the small-x behavior of the forward distribution is an increasing function of Q^2. As a result, for fixed b, the $\mathcal{R}^{(b)}(\zeta)$ ratio increases with Q^2. In general, the profile of $\tilde{f}(\tilde{x},\alpha)$ in the α-direction is also affected by the pQCD evolution. In particular, in ref. [6] it was shown that if one takes an ansatz corresponding to an extremely asymmetric profile function $\rho(\alpha) \sim \delta(1+\alpha)$, the shift of the profile function to a more symmetric shape is clearly visible in the evolution of the relevant SPD. In the next section, we will study the interplay between evolution of x and α profiles of DDs.

V. QCD EVOLUTION AND PROFILE OF DDS

Both the shape of the forward distributions $\tilde{f}(x;\mu)$ reflected in the x-dependence of the DDs $\tilde{f}(x,\alpha;\mu)$ and their profile in the α-direction are affected by the pQCD evolution. At the one-loop level, the solution for QCD evolution equations is known in the operator form [18], so that choosing specific matrix elements one can convert the universal solution into four (at least) evolution patterns: for usual parton densities ($\langle p|\dots|p\rangle$ case), distribution amplitudes ($\langle 0|\dots|p\rangle$ case), skewed and double parton distributions ($\langle p-r|\dots|p\rangle$ case). Since all the types of the pQCD evolution originate from the same source, one may expect an interplay between the x- and α- aspects of the DDs evolution.

In the simplest case of flavor-nonsinglet (valence) functions, the multiplicatively renormalizable operators were originally found in Ref. [19]

$$\mathcal{O}_n^{NS} = (z\partial_+)^n\,\bar{\psi}\lambda^a\not{z}C_n^{3/2}(z\overleftrightarrow{D}/z\partial_+)\psi\,. \tag{5.1}$$

In contrast, the usual operators $\bar{\psi}\lambda^a\not{z}(z\overleftrightarrow{D})^n\psi$ mix under renormalization with the lower spin operators $(z\partial_+)^{n-k}\bar{\psi}\lambda^a\not{z}(z\overleftrightarrow{D})^k\psi$. The symbolic notation $(z\overleftrightarrow{D}/z\partial_+)$ with $\overleftrightarrow{D} \equiv \overrightarrow{D} - \overleftarrow{D}$, $\partial_+ \equiv \overrightarrow{D} + \overleftarrow{D} = \overrightarrow{\partial} + \overleftarrow{\partial}$ and $C_n^{3/2}(\alpha)$

being the Gegenbauer polynomials is borrowed from Ref. [19]. In Ref. [19] it was also noted that these operators coincide with the free-field conformal tensors. As pointed out in Ref. [3], the multiplicative renormalizability of $\mathcal{O}_n^{NS}$ operators means that the Gegenbauer moments

$$\mathcal{C}_n^{NS}(\xi,\mu) = \xi^n \int_{-1}^{1} C_n^{3/2}(z/\xi)\, H^{NS}(z,\xi;\mu)\, dz \tag{5.2}$$

of the skewed parton distribution $H^{NS}(z,\xi;\mu)$ have a simple evolution [3]:

$$\mathcal{C}_n^{NS}(\xi,\mu) = \mathcal{C}_n^{NS}(\xi,\mu_0)\left[\frac{\ln\mu_0/\Lambda}{\ln\mu/\Lambda}\right]^{\gamma_n/\beta_0}, \tag{5.3}$$

where $\beta_0 = 11 - \frac{2}{3}N_f$ is the lowest coefficient of the QCD β-function and γ_n's are the nonsinglet anomalous dimensions [20,21]. Going from SPDs to DDs, writing the SPD variable $\tilde{x}$ in terms of DD variables $\tilde{x} = x + \alpha\xi$ and using

$$C_n^{3/2}(x/\xi + \alpha) = \sum_{l=0}^{n} \frac{\Gamma(n-l+3/2)}{\Gamma(3/2)(n-l)!}\,(2x/\xi)^{n-l}\, C_l^{3/2+n-l}(\alpha)\,, \tag{5.4}$$

one can express the Gegenbauer moments $\mathcal{C}_n(\xi,\mu)$ in terms of the combined [x-ordinary $\otimes$ α-Gegenbauer] moments of the relevant DDs:

$$\mathcal{C}_n^{NS}(\xi,\mu) = \sum_{k=0}^{[n/2]} \xi^{2k} \int_{-1}^{1} dx \int_{-1+|x|}^{1-|x|} 2^{n-2k}\, \frac{\Gamma(n-2k+3/2)}{\Gamma(3/2)(n-2k)!}\, x^{n-2k} C_{2k}^{3/2+n-2k}(\alpha)\; \tilde{f}^{NS}(x,\alpha;\mu)\, d\alpha\,. \tag{5.5}$$

Hence, each $x^m C_l^{3/2+m}(\alpha)$ moment of $\tilde{f}^{NS}(x,\alpha;\mu)$ is multiplicatively renormalizable and its evolution is governed by the anomalous dimension γ_{l+m} [3,6]. In Eq. (5.5), we took into account that α-DDs $\tilde{f}(x,\alpha)$ are always even in α, which gives an expansion of the Gegenbauer moments in powers of ξ^2. In the nonsinglet case, the Gegenbauer moments $\mathcal{C}_n(\xi,\mu)$ are nonzero for even n only. A similar represenation can be written for the Gegenbauer moments of the singlet quark distributions. In the latter case, the DD $\tilde{f}^S(x,\alpha)$ is odd in x, and only odd Gegenbauer moments $\mathcal{C}_n^S(\xi,\mu)$ do not vanish.

Another simple case is the evolution of the gluon distributions in pure gluodynamics. Then the multiplicatively renormalizable operators with the same Lorentz spin $n+1$ as in Eq. (5.1) are

$$\mathcal{O}_n^G = z^\mu z^\nu (z\partial_+)^{n-1} G_{\mu\alpha} C_{n-1}^{5/2}(z\overleftrightarrow{D}/z\partial_+)\, G_{\alpha\nu}\,. \tag{5.6}$$

Due to the symmetry properties of gluon DDs, only Gegenbauer moments

$$\mathcal{C}_n^G(\xi,\mu) = \xi^{n-1} \int_{-1}^{1} C_{n-1}^{5/2}(z/\xi)\, H^G(z,\xi;\mu)\, dz \tag{5.7}$$

with odd n do not vanish. The Gegenbauer moment can also be written in terms of DDs:

$$\mathcal{C}_n^G(\xi,\mu) = \sum_{k=0}^{[(n-1)/2]} \xi^{2k} \int_{-1}^{1} dx \int_{-1+|x|}^{1-|x|} 2^{n-2k-1}\, \frac{\Gamma(n-2k+3/2)}{\Gamma(5/2)(n-2k-1)!}\, x^{n-2k} C_{2k}^{3/2+n-2k}(\alpha)\; \tilde{f}^G(x,\alpha)\, d\alpha\,. \tag{5.8}$$

Two shifts: $n \to n-1$ and $3/2 \to 5/2$ in some sense compensate each other. Again, each combined $x^m C_l^{3/2+m}(\alpha)$ moment of $\tilde{f}^G(x,\alpha)$ is multiplicatively renormalizable and its evolution is governed by the anomalous dimension γ_{l+m}^{GG} [3,6].

Since the Gegenbauer polynomials $C_l^{3/2+m}(\alpha)$ are orthogonal with the weight $(1-\alpha^2)^{m+1}$, evolution of the x^m-moments of DDs in both cases is given by the formula [6]

$$\tilde{f}_m(\alpha\,;\mu) \equiv \int_{-1}^{1} x^m \tilde{f}(x,\alpha\,;\mu)\, dx = (1-\alpha^2)^{m+1} \sum_{k=0}^{\infty} A_{ml} C_l^{m+3/2}(\alpha)\, [\log(\mu/\Lambda)]^{-\gamma_{m+l}/\beta_0}\,, \tag{5.9}$$

where the coefficients A_{ml} are proportional to $x^m C_l^{3/2+m}(\alpha)$ moments of DDs. A similar representation holds in the singlet case, with $[\log(\mu/\Lambda)]^{-\gamma_{m+l}/\beta_0}$ substituted by a linear combination of terms involving $[\log(\mu/\Lambda)]^{-\gamma_{m+l}^+/\beta_0}$ and $[\log(\mu/\Lambda)]^{-\gamma_{m+l}^-/\beta_0}$ with singlet anomalous dimensions $\gamma_{m+l}^{\pm}$ obtained by diagonalizing the coupled quark-gluon evolution equations [6].

The anomalous dimensions γ_n increase with raising n, and, hence, the mth x-moment of $\tilde{f}(x,\alpha;\mu)$ is asymptotically dominated by the α-profile $(1-\alpha^2)^{m+1}$. Such a correlation between x- and α-dependences of $\tilde{f}(x,\alpha;\mu)$ is not something exotic. Take a DD which is constant in its support region. Then its x^m-moment behaves like $(1-|\alpha|)^{m+1}$, i.e., the width of the α profile decreases with increasing n. This result is easy to understand: due to the spectral condition $|\alpha| \le 1-|x|$, the x^m moments with larger m are dominated by regions which are narrower in the α-direction.

These observations suggests to try a model in which the moments $\tilde{f}_m(\alpha;\mu)$ has the asymptotic $(1-\alpha^2)^{m+1}$ profile even at non-asymptotic μ. This is equivalent to assuming that all the combined moments $x^m C_l^{3/2+m}(\alpha)$ with $l > 0$ vanish. Note that this assumption is stable wrt pQCD evolution. Since integrating $\tilde{f}_m(\alpha\,;\mu)$ over α one should get the moments $\tilde{f}_m(\mu)$ of the forward density $f(x;\mu)$, the DD moments $\tilde{f}_m(\alpha;\mu)$ in this model are given by

$$\tilde{f}_m(\alpha;\mu) = \rho_{m+1}(\alpha)\tilde{f}_m(\mu) \tag{5.10}$$

where $\rho_{m+1}(\alpha)$ is the normalized profile function (cf. Eq.(4.5)). In explicit form:

$$\int_{-1}^{1} x^m \tilde{f}(x,\alpha\,;\mu)\, dx = \frac{\Gamma(m+5/2)}{\Gamma(1/2)\,(m+1)!}(1-\alpha^2)^{m+1} \int_{-1}^{1} \tilde{f}(x;\mu) x^m dx\,. \tag{5.11}$$

In this relation, all the dependence on α can be trivially shifted to the lhs of this equation, and we immediately see that $\tilde{f}(x,\alpha\,;\mu)$ in this model is a function of $x/(1-\alpha^2)$:

$$\tilde{f}(x,\alpha\,;\mu) = F(x/(1-\alpha^2);\,\mu)\,\theta(0 < x/(1-\alpha^2) < 1)\,. \tag{5.12}$$

A direct relation between $\tilde{f}(z,\mu)$ and $F(u;\mu)$ can be easily obtained using the basic fact that integrating $\tilde{f}(x,\alpha\,;\mu)$ over α one should get the forward density $\tilde{f}(z,\mu)$; e.g., for positive z we have

$$f(z) = z\int_z^1 \frac{F(u)}{u^{3/2}\sqrt{u-z}}\, du\,. \tag{5.13}$$

This relation has the structure of the Abel equation. Solving it for $F(u)$ we get

$$F(u) = -\frac{u^{3/2}}{\pi}\int_u^1 \frac{[f(z)/z]'}{\sqrt{z-u}}\, dz\,. \tag{5.14}$$

Thus, in this model, knowing the forward density $f(z)$ one can calculate the double distribution function $\tilde{f}(x,\alpha) = F(x/(1-\alpha^2))$.

Note, however, that the model derived above violates the DD support condition $|x| + |\alpha| \le 1$: the restriction $|x| \le 1-\alpha^2$ defines a larger area. Hence, the model is only applicable in a situation when the difference between two spectral conditions can be neglected. A practically important case is the shape of $H(\tilde{x},\xi)$ for small ξ. Indeed, calculating $H(\tilde{x},\xi)$ for small ξ one integrates the relevant DDs $\tilde{f}(\tilde{x})$ over practically vertical lines. If $\tilde{x}$ is also small, both the correct $|\alpha| \le 1-|x|$ and model $\alpha^2 \le 1-|x|$ conditions can be substituted by $|\alpha| \le 1$. Now, if $\tilde{x} \gg \xi$, a slight deviation of the integration line from the vertical direction can be neglected and $H(\tilde{x},\xi)$ can be approximated by the forward limit $\tilde{f}(\tilde{x})$.

Specifying the ansatz for $f(z)$, one can get an explicit expression for the model DD by calculating $F(u)$ from Eq. (5.14). However, in the simplest case when $f(x) = Ax^{-a}$ for small x, the result is evident without any calculation:

the DD $f(x,\alpha)$ which is a function of the ratio $x/(1-\alpha^2)$ and reduces to Ax^{-a} after in integration over α must be given by

$$f(x,\alpha) = \rho_a(\alpha) f(x)$$

where $\rho_a(\alpha)$ is the normalized profile function of Eq.(4.5):

$$f(x,\alpha) = A\,\frac{\Gamma(a+5/2)}{\Gamma(1/2)\,\Gamma(a+2)}(1-\alpha^2)^a\,x^{-a}. \tag{5.15}$$

This DD is a particular case of the general factorized ansatz $f(x,\alpha) = \rho_n(\alpha) f(x)$ considered in the previous section. Its most nontrivial feature is the correlation $n = a$ between the profile function parameter n and the power a characterizing the small-x behavior of the forward distribution.

Knowing DDs, the relevant SPDs $H(\tilde{x},\xi)$ can be obtained in the standard way from $\tilde{f}(x,\alpha)$ for quarks and from $x\tilde{f}^G(x,\alpha)$ in the case of gluons. In particular, the SPD enhancement factor $\mathcal{R}(\zeta)$ for small ζ in this model is given by

$$\frac{\mathcal{F}_\zeta^Q(\zeta)}{f^Q(\zeta)} = \frac{\Gamma(2a+2)}{\Gamma(a+2)\Gamma(a+1)} \tag{5.16}$$

for quarks and by

$$\frac{\mathcal{F}_\zeta^G(\zeta)}{\zeta f^G(\zeta)} = \frac{\Gamma(2a+2)}{\Gamma(a+3)\Gamma(a+1)} \tag{5.17}$$

for gluons.

The use of the asymptotic profiles for DD moments $\tilde{f}_n(\alpha)$ is the basic assumption of the model described above. However, if one is interested in SPDs for small ξ, the impact of deviations of $\tilde{f}_n(\alpha)$ from the asymptotic profile is suppressed. Even if the higher harmonics are present in $\tilde{f}_n(\alpha)$, i.e., if the $x^{n-2k}C_{2k}^{3/2+n-2k}(\alpha)$ moments of $\tilde{f}(x,\alpha)$ are nonzero for $k \geq 1$ values, their contribution into the Gegenbauer moments $C_n(\xi,\mu)$ is strongly suppressed by ξ^{2k} factors [see Eq.(5.5)]. Hence, for small ξ, the shape of $H(\tilde{x},\xi)$ for a wide variety of model α-profiles is very close to that based on the asymptotic profile model.

Absence of higher harmonics in $\tilde{f}_n(\alpha)$ is equivalent to absence of the ξ-dependence in the Gegenbauer moments $C_n(\xi,\mu)$. The assumption that $C_n(\xi,\mu)$ do not depend on ξ was the starting point for the model of SPDs $H(\tilde{x},\xi)$ constructed in ref. [8]. Though the formalism of DDs was not used in ref. [8], both approaches lead to identical results: the final result of [8] has the form of a DD representation for $H(\tilde{x},\xi)$. In Appendix A, we also start with $C_n(\xi,\mu) = C_n(0,\mu)$ and rederive the DD corresponding to the asymptotic profile model.

VI. EVOLUTION ALGORITHM

At one loop, evolution equations for nonforward parton distributions $\mathcal{F}_\zeta^a(x,\mu)$ can be written as

$$\mu\frac{d\mathcal{F}_\zeta^a(x,\mu)}{d\mu} = \sum_b \int_0^1 W_\zeta^{ab}(x,z)\mathcal{F}_\zeta^b(z,\mu^2)\,dz \tag{6.1}$$

or in a "matrix" notation

$$\mu\frac{d\hat{\mathcal{F}}_\zeta^a(\mu)}{d\mu} = \hat{W}_\zeta^{ab} \otimes \hat{\mathcal{F}}_\zeta^b(\mu)\,. \tag{6.2}$$

Using the explicit one-loop form of the effective coupling constant

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \log(\mu^2/\Lambda^2)} \tag{6.3}$$

($\beta_0 = 11 - \frac{2}{3}N_f$) and the symbolic notations of (6.2), one can present the formal solution for the set of evolution equations in the form of an expansion

$$\hat{\mathcal{F}}_\zeta(Q^2) = \exp\left(L(Q^2,Q_0{}^2)\,\hat{W}_\zeta\right) \otimes \hat{\mathcal{F}}_\zeta(Q_0{}^2) \tag{6.4}$$

$$= \sum_n \frac{(L(Q^2,Q_0{}^2)^n}{n!}\left[\left(\hat{W}_\zeta\right)^n \otimes \hat{\mathcal{F}}_\zeta(Q_0{}^2)\right]. \tag{6.5}$$

To get the $n = 1$ term $\delta_1\hat{\mathcal{F}}_\zeta \equiv \hat{W}_\zeta \otimes \hat{\mathcal{F}}_\zeta(Q_0{}^2)$ of this expansion, we evaluate numerically the convolution of the kernel $\hat{W}_\zeta$ with the initial distributions $\hat{\mathcal{F}}_\zeta(Q_0{}^2)$. To get the $n = 2$ term $\delta_2\hat{\mathcal{F}}_\zeta \equiv (\hat{W}_\zeta)^2 \otimes \hat{\mathcal{F}}_\zeta(Q_0{}^2)$, we convolute $\hat{W}_\zeta$ with the smoothly interpolated result of the first iteration $\delta_1\hat{\mathcal{F}}_\zeta$, and so on. After obtaining $\delta_1\hat{\mathcal{F}}_\zeta$, $\delta_2\hat{\mathcal{F}}_\zeta$, etc., we construct evolved distributions $\hat{\mathcal{F}}_\zeta(Q^2)$ for any desired value of Q^2. Of course, the number of necessary iterations of $\hat{W}_\zeta$ with the initial distributions $\hat{\mathcal{F}}_\zeta(Q_0{}^2)$ depends on the size of the expansion parameter

$$L(Q^2,Q_0{}^2) = \frac{2}{\beta_0}\log\left[\frac{\log\left(Q^2/\Lambda^2\right)}{\log\left(Q_0{}^2/\Lambda^2\right)}\right]. \tag{6.6}$$

When L is not very large, it is sufficient to calculate just one or two iterations.

VII. EVOLUTION OF NONFORWARD QUARK DISTRIBUTIONS

The evolution of skewed parton distributions was studied numerically in refs. [15,16,12,14,17]. In this section, we perform the numerical evolution of SPDs using the algorithm described in the previous session and present the results illustrating evolution patterns for SPDs constructed using the factorized model (3.1) with different choices for the profile function $h(x,\alpha)$.

As we discussed earlier, the use of the infinitely narrow profile function $h(x,\alpha) = \delta(\alpha)$ gives the simplest model in which OFPDs $H(\tilde{x},\xi)$ coincide with forward distributions $\tilde{f}(\tilde{x})$. In terms of NFPDs, this modell looks less trivial. It gives $F^{(\infty)}(x,y) = \delta(y-\tilde{x}/2)f(x)$ for the y-DDs which results in (untilded) NFPDs given by shifted forward distributions

$$\mathcal{F}_\zeta(X) = \frac{1}{(1-\zeta/2)}f\left(\frac{X-\zeta/2}{1-\zeta/2}\right).$$

For any monotonic function $f(x)$ this gives NFPDs $\mathcal{F}_\zeta(X)$ which are larger in the region $X \geq \zeta$ than their forward counterparts. Due to pQCD evolution, $f(x,Q^2)$ get steeper in small-x region, i.e. the NFPDs become even more strongly enhanced.

As noted in Section IV, the use of wide profile functions results in stronger enhancement for NFPDs in $X \geq \zeta$ region. For llustration see Figs.6,7. In Fig. 6 we show $\zeta = 0.1$ NFPDs obtained "by shift" from forward distributions $f(x,Q^2)$ taken at two Q^2 values 1.5 GeV^2 and 20 GeV^2. For comparison, we show also NFPDs obtained from $f(x,Q^2 = 1.5\,\mathrm{GeV}^2)$ using the y-DDs with "asymptotic" profile functions $F_{as}^{NS}(x,y) = 6y(1-x-y)f^{NS}(x)/(1-x)^3$ for nonsinglet quark distributions and $F_{as}^{YM}(x,y) = 30y^2(1-x-y)^2/(1-x)^5 f^G(x)$ for gluon distibutions in pure gluodynamics. Specifically, we took $f^{NS}(x,Q^2 = 1.5\,\mathrm{GeV}^2) = \frac{35}{32}x^{-1/2}(1-x)^3$and $f^G(x,Q^2 = 1.5\,\mathrm{GeV}^2) = 0.4x^{-0.3}\ln(1/x)(1-x)^5$. The NFPDS constructed in this way were then numerically evolved to $Q^2 = 20\,\mathrm{GeV}^2$ using the approach outlined in Section VI and kernels given Appendix B. The ratio of NFPDs obtained using these two models is shown in Fig.7. As expected, the ratios increase with Q^2.

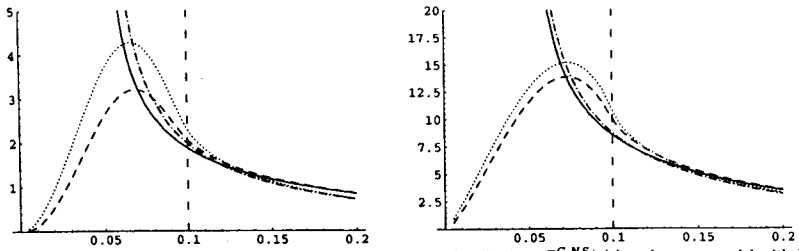


FIG. 6. YM gluon (left) and quark nonsinglet (right) nonforward distributions $F_\zeta^{G,NS}(x)$ based on our model with h_{as} profiles versus "shifted" forward ones $\frac{1}{(1-\zeta/2)} f\left(\frac{x-\zeta/2}{1-\zeta/2}\right)$, at two different Q^2 scales: $1.5\,\mathrm{GeV}^2$ (dashed and solid curves) and $20\mathrm{GeV}^2$ (dotted and dash-dotted curves).

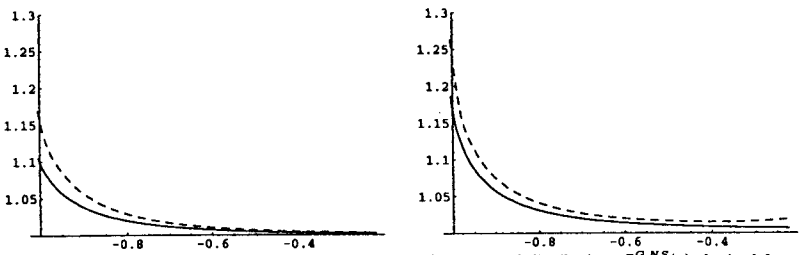


FIG. 7. Ratio of YM gluon (left) and nonsinglet quark (right) nonforward distributions $F_\zeta^{G,NS}(x)$ obtained from our model with h_{as} profiles to the "shifted" forward ones $\frac{1}{(1-\zeta/2)} f\left(\frac{x-\zeta/2}{1-\zeta/2}\right)$, at two different Q^2 scales: $1.5\,\mathrm{GeV}^2$ (solid) and $20\,GeV^2$ (dashed); $\zeta = 0.1$.

The same construction can be performed in the singlet case. Main observation here is large enhancement for singlet quark distributions at $Q^2 = 20\,\mathrm{GeV}^2$, with the $\mathcal{F}_\zeta^S(\zeta)/f^S(\zeta/2)$ ratio being close to 1.8, see Fig.8. This is again in agreement with the estimates made in Section IV for $a \approx 1$.

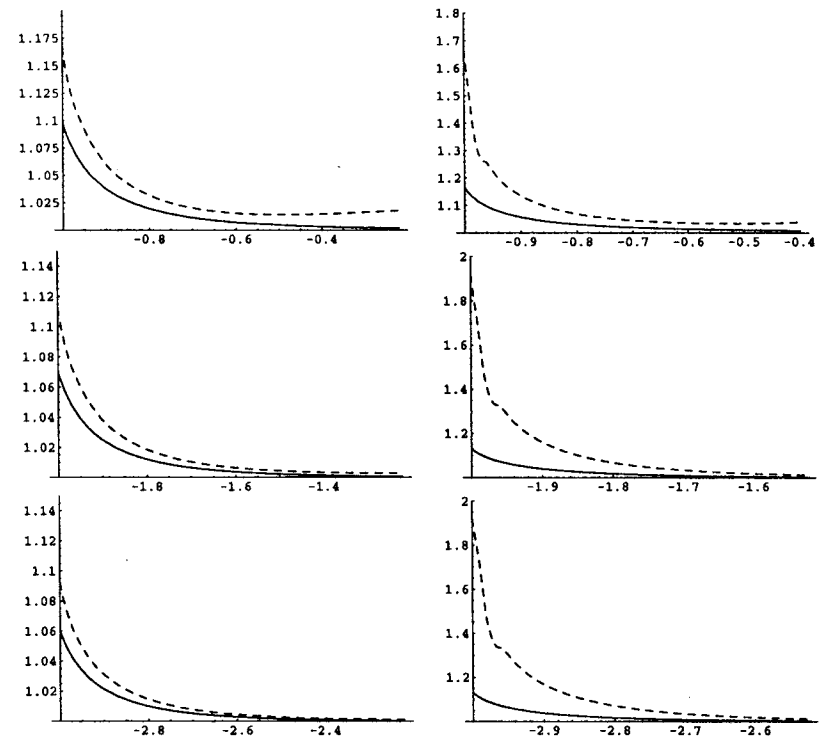


FIG. 8. Ratio of QCD gluon (left) and singlet quark (right) nonforward distributions $F_\zeta^{G,S}(x)$ obtained from our model with h_{as} profiles to the "shifted" forward ones $\frac{1}{(1-\zeta/2)} f\left(\frac{x-\zeta/2}{1-\zeta/2}\right)$, at two different Q^2 scales: $1.5\,\mathrm{GeV}^2$ (solid) and $20\,\mathrm{GeV}^2$ (dashed); $\zeta = 0.1$ - top, $\zeta = 0.01$ - middle, $\zeta = 0.001$ - bottom.

In the above examples of nonsinglet quark distributions and gluon distributions in pure gluodynamics we took "asymptotic" profiles. It is interesting to test whether these profiles are really stable under pQCD evolution. To this end, we compared two models for $Q^2 = 20\,\mathrm{GeV}^2$ distributions. First, we took the forward distribution evolved to $Q^2 = 20\,\mathrm{GeV}^2$ and constructed model NFPD using the "asymptotic" profile. Second way is to construct NFPD from $Q^2 = 1.5\,\mathrm{GeV}^2$ forward distribution using the asymptotic profile and then evolve NFPD to $Q^2 = 20\,\mathrm{GeV}^2$ using nonforward kernels. Fig. 9 shows that the results obtained in the two ways are practically identical.

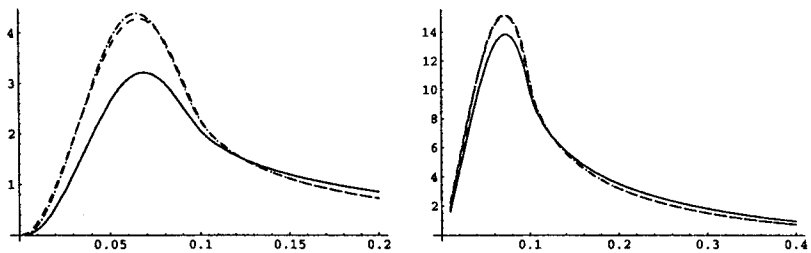
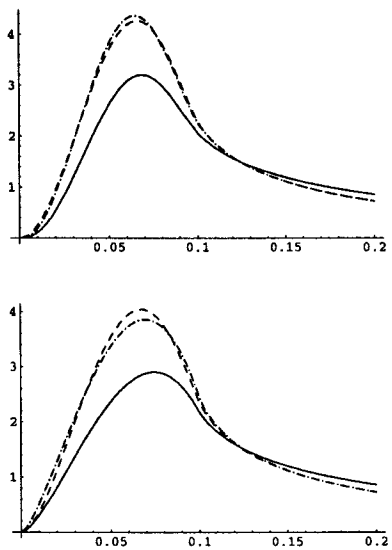


FIG. 9. Evolution of the YM gluon (left) and nonsinglet quark (right) nonforward distributions $F_\zeta^{G,q}(x)$ obtained in our model with h_{as} profiles. Solid curves correspond to the initial distributions at $Q^2 = 1.5\,\mathrm{GeV}^2$. Dashed curves represent nonforward distributions evolved to $Q^2 = 20\,\mathrm{GeV}^2$, dash-dotted ones are obtained from the model with the same h_{as} profiles and with forward distributions evolved to $Q^2 = 20\,\mathrm{GeV}^2$.

However, if one takes profiles strongly differing from the "asymptotic" ones, the curves obtained in the two ways described above, visibly differ from each other, see Fig.10. In the case of a wide profile function, the evolved NFPD looks like that constructed from evolved forward distribution but using a narrower profile. In other words, the pQCD evolution in this case narrows the profile function. Alternatively, if one starts with a too narrow profile, then the evolved NFPD resembles the model function constructed from evolved forward distribution but using a wider profile.



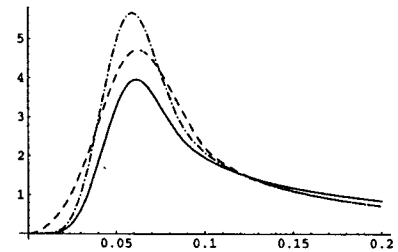
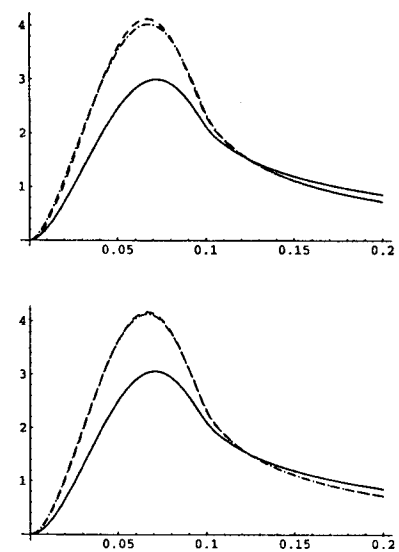


FIG. 10. The same as on previous set of figures (gluon distributions only) but with different profiles $h(x,y) = N_\lambda(x)(y(1-x-y))^\lambda$: asymptotic ($\lambda = 2$) - top, wide ($\lambda = 1$) - middle, narrow ($\lambda = 6$) - bottom.

The study performed in Section V (see also Appendix A) demonstrated that at large Q^2 there should be a correlation between the x-dependence of the forward distributions and the form of the profile function. Taking the GRV-type parametrization for gluon (with $f_g(x) \sim x^{-0.3}\ln(1/x)$) and quark singlet distributions at $Q^2 = 1.5\mathrm{GeV}^2$, we again compared the $Q^2 = 20\mathrm{GeV}^2$ curves constructed in two ways described above. A better agreement between the two models was observed for $\lambda = 1.5$ rather than for $\lambda = 1.3$. However, the $\lambda = 1.3$ profile works perfectly if one takes the model with purely power-like behavior of the gluon distribution $f_g(x) \sim x^{-0.3}$, see Fig.11.



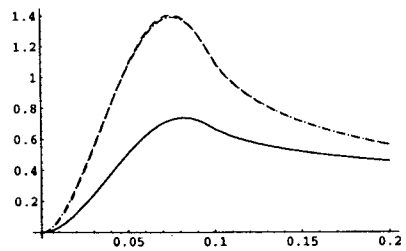


FIG. 11. Top and middle: the same as on previous set of figures but with other profiles $h(x,y) = N_\lambda(x)(y(1-x-y))^\lambda$: $\lambda = 1.3$ - top, $\lambda = 1.5$ - middle. Bottom figure represents the evolution of the model obtained with the profile parameter $\lambda = 1.3$ and model forward distribution $f_G(x) = 1/x^{0.3}$ (no logarithm).

In the singlet case, it is more convenient to use "tilded" distributions defined on the $[-1+\zeta \le X \le 1]$ segment. These functions, shown in Fig.12, are symmetric or antisymmetric wrt the middle point $X = \zeta/2$.

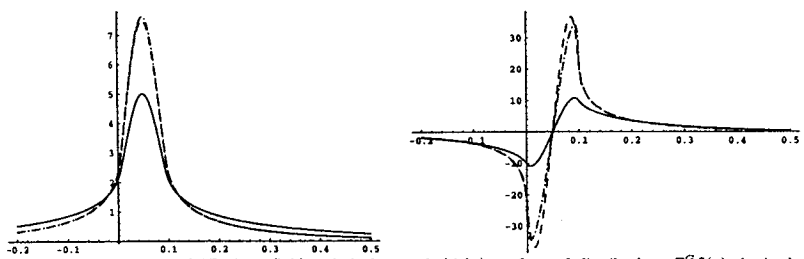


FIG. 12. Evolution of the QCD gluon (left) and singlet quark (right) nonforward distributions $F_\zeta^{G,q}(x)$ obtained our model with h_{as} profiles. Solid curves correspond to the initial distributions at $Q^2 = 1.5\,\mathrm{GeV}^2$. Dashed curves represent nonforward distributions evolved to $Q^2 = 20GeV^2$, dash-dotted ones are obtained from the model with the same h_{as} profiles and with forward distributions evolved to $Q^2 = 20\,\mathrm{GeV}^2$.

VIII. CONCLUSIONS

In this paper, we discussed the structure of double and skewed distributions and their relation to the usual (forward) parton distributions. We emphasized that there are meson-exchange type terms in DDs and SPDs which are invisible in the forward limit. Theoretically, they can be modeled within approaches involving the mesons as elementary fields, e.g., this has been done in ref. [9] within the chiral soliton model framework. There are also terms in SPDs and DDs which are related to usual distributions through reduction relations and which are, in this sense, visible from the forward limit. It is just these "forward visible" parts of DDs and SPDs which are the subject of our studies in the present paper. We proposed factorized models for double distributions in which one factor coincides with the usual (forward) parton distribution and another specifies the profile characterizing the spread of the longitudinal momentum transfer. By construction, these models satisfy reduction formulas. Then we used the factorized model DDs to construct skewed parton distributions and studied the skewedness dependence of SPDs. We established that, for small skewedness, the relevant parts of SPDs $H(\tilde{x},\xi)$ can be obtained by averaging the usual parton densities $f(\tilde{x}-\xi\alpha)$ with the weight $\rho(\alpha)$ coinciding with the profile function of the double distribution $\tilde{f}(x,\alpha)$ at small x. This result allows one to get estimates for the ratio of SPDs taken at the border point $\tilde{x} = \xi$

(or $X = \zeta$) and usual parton densities taken at $x = x_{Bj}$. This ratio is larger than 1 in all cases which we have considered, i.e., SPDs in this sense are enhanced compared to the forward densities. We found that, for small x_{Bj}, the enhancement factor is directly related to the parameter a characterizing the effective power behavior x^{-a} of the usual parton densities. We gave an explicit expression for the enhancement factor which involves a and the parameter b characterizing the effective power profile $(1-\alpha^2)^b$ of the relevant DD. Furthermore, we demonstrated that if the x^n moments $\tilde{f}_n(\alpha)$ of the α-DDs have the asymptotic $(1-\alpha^2)^{n+1}$ profile, then the parameters a and b are correlated: $b = a$, i.e., the α-profile of $\tilde{f}(x,\alpha)$ for small x is completely determined in this case by the small-x behavior of the usual parton distribution. We also demonstrated that in the case of small skewedness ξ, the deviation of $\tilde{f}_n(\alpha)$ from its asymptotic form leads to very small $O(\xi^2)$ corrections only. This means that the "forward visible" parts of SPDs can be rather accurately deduced from the purely forward usual parton densities. The absence of higher harmonics in $\tilde{f}_n(\alpha)$ is equivalent to the absence of ξ-dependence in the Gegenbauer moments of SPDs, which is the starting point of the model for SPDs constructed in the recent paper by Shuvaev et al. [8]. For this reason, the results based on the asymptotic profile model (APM) developed in the present paper coincide with those of ref. [8], though its authors refrained from using DDs in their approach. Finally, we performed a numerical investigation of the evolution patterns of SPDs and gave interpretation of the results of these studies within the DD framework. In particular, if the initial profile of DDs is too wide, the pQCD evolution makes it more narrow and the profile widens if initially it was too narrow. Our numerical results also support the expectation that if one takes SPD derived from the forward distribution by APM prescription and evolves it using nonforward evolution kernels, the result coincides with SPD which is APM-derived from evolved forward distribution. Summarizing, our results based on numerical evolution of SPDs are in full accord with theoretical expectations based on DD formalism. The next step in this investigation should be a direct numerical evolution of double distributions.

ACKNOWLEDGMENTS

We thank I. Balitsky, A. Belitsky, V. Braun, C. Coriano, L. Frankfurt, A. Freund, K. Golec-Biernat, L. Gordon, I. Grigentch, V. Guzey, X. Ji, G. Korchemsky, A. Martin, L. Mankiewicz, D. Müller, G. Piller, M. Polyakov, M. Ryskin, A. Schäfer, A. Shuvaev, M. Strikman and C. Weiss for stimulating discussions and correspondence. This work was supported by the US Department of Energy under contract DE-AC05-84ER40150.

APPENDIX A: ASYMPTOTIC PROFILE MODEL AND ξ-INDEPENDENT GEGENBAUER MOMENTS

The lowest $k = 0$ term of the ξ^{2k} expansion for the Gegenbauer moments of the SPDs involves the DDs $\tilde{f}(x,\alpha)$ integrated over α with $C_0^{n+3/2}(\alpha) = 1$, i.e. the ordinary (forward) parton distributions:

$$\mathcal{C}_n^Q(\xi=0,\mu) = 2^n \frac{\Gamma(n+3/2)}{\Gamma(3/2)\,n!} \int_{-1}^{1} \tilde{f}^Q(x;\mu)\, x^n dx \,. \tag{A1}$$

$$\mathcal{C}_n^G(\xi=0,\mu) = 2^n \frac{\Gamma(n+3/2)}{3\,\Gamma(3/2)\,(n-1)!} \int_{-1}^{1} \tilde{f}^Q(x;\mu)\, x^n dx \,. \tag{A2}$$

The higher terms of the ξ^{2k} expansion are small for small ξ. The approximation in which the $k \neq 0$ terms of the ξ^{2k} expansion for the Gegenbauer moments are neglected (i.e., Gegenbauer moments $\mathcal{C}_n(\xi,\mu)$ are treated as ξ-independent) can be translated into a model for the double distribution $\tilde{f}(x,\alpha)$. The advantage of this model is that its basic assumption is stable with respect to evolution: if the Gegenbauer moments $\mathcal{C}_n(\xi,\mu_0)$ are ξ-independent at some low normalization point μ_0, this property is preserved for any higher μ. The only impact of the pQCD evolution is the change of the relevant forward parton distribution from $f(x,\mu_0)$ to $f(x,\mu)$.

To derive the explicit expression for $\tilde{f}(x,\alpha)$ in this model, we will use the expansion of the light-cone operator $\bar{\psi}(-z/2)\lambda^a \hat{z}\psi(z/2)$ over the multiplicatively renormalizable operators $\mathcal{O}_n$ (see [18])

$$\bar{\psi}(-z/2)\lambda^a \hat{z}\psi(z/2) = \sum_{n=0}^{\infty}(-1)^n \frac{(2n+3)}{4^{n+1}(n+1)!}\int_{-1}^{1}(1-\alpha^2)^{n+1}\mathcal{O}_n(\alpha z/2)\,d\alpha\,. \tag{A3}$$

Inserting it into the nonforward matrix element, we obtain

$$\langle P-r/2\,|\,\bar{\psi}(-z/2)\lambda^a \hat{z}\psi(z/2)\,|\,P+r/2\rangle = \int_{-1}^{1} e^{-i\alpha(rz)/2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+3)}{2^{n+2}(n+1)!}(1-\alpha^2)^{n+1}(-iPz)^n\mathcal{C}_n(\xi,\mu)\,d\alpha\,. \tag{A4}$$

If the Gegenbauer moments $\mathcal{C}_n(\xi,\mu)$ are approximated by their $\xi=0$ values, Eq.(A4) can be transformed into the representation of the nonforward matrix element in terms of double distributions. Namely, the nth moment

$$\tilde{f}_n(\alpha\,;\mu) = \int_{-1}^{1} x^n \tilde{f}(x,\alpha\,;\mu)\,dx \tag{A5}$$

of the α-DD $\tilde{f}(x,\alpha\,;\mu)$ is then given by

$$\tilde{f}_n(\alpha\,;\mu) = \frac{\Gamma(n+5/2)}{\Gamma(1/2)\,(n+1)!}(1-\alpha^2)^{n+1}\int_{-1}^{1} f(z;\mu)z^n dz\,. \tag{A6}$$

The factor relating $\tilde{f}_n(\alpha\,;\mu)$ and $\tilde{f}_n(\mu)$ is just the normalized profile function $\rho_{n+1}(\alpha)$ (see Section ?). In the nonsinglet case only moments with even n are nonzero, while in the singlet case only those with odd n do not vanish. Below, we will construct the $x>0$ parts of DDs, the trivial (anti)symmetrization can be performed at the end.

Incorporating the inverse Mellin transformation, one can obtain the kernel $K(x,\alpha;z)$ which relates DDs $f(x,\alpha)$ in this model with the usual forward distributions $f(z)$. However, inverting the representation (A6) "as is", one would get the expression

$$f(x,\alpha) \stackrel{?}{=} -\frac{1}{2\pi}\int\limits_{x/(1-\alpha^2)}^{1}\left[z(1-\alpha^2)/x-1\right]^{-3/2} f(z)\,\frac{dz}{z}\,, \tag{A7}$$

whose rhs has a suspicious overall sign. Furthermore, the integral over z diverges at the end-point $z=x/(1-\alpha^2)$. These inconsistencies indicate that the implied interchange of the inverse Mellin transformation and the z-integral is not justified. To get a less singular kernel, one can try to add an $O(1/n)$ factor in the expression for the moments $f_n(\alpha\,;\mu)$, e.g., convert $1/(n+1)!$ into $1/(n+2)!$ (after such a change, the inverse Mellin transform is still easily doable analytically). To this end, we use the relation

$$\int_0^1 f(z;\mu)z^n dz = -\frac{1}{n+2}\int_0^1 z^{n+2}\frac{d}{dz}\left(\frac{f(z;\mu)}{z}\right)dz \tag{A8}$$

which holds if (a) the function $f(z;\mu)$ vanishes at $z=1$ (this is always true) and (b) $z^{n+1}f(z)$ vanishes at $z=0$. The latter requirement is evidently satisfied in the nonsinglet case for all $n\geq 0$. The singlet distributions are more singular, but we need only their $n\geq 1$ moments, i.e., the restriction (b) is satisfied again. Using Eq.(A8) produces the kernel which connects $f(x,\alpha)$ with $[f(z)/z]'$:

$$f(x,\alpha) = -\frac{x}{\pi(1-\alpha^2)}\int\limits_{x/(1-\alpha^2)}^{1}\left[z(1-\alpha^2)/x-1\right]^{-1/2}[f(z)/z]'\,dz\,. \tag{A9}$$

This result coincides with Eq.(5.14). The spectral condition $x/z\leq 1-\alpha^2$ relating the "original" fraction z and the "produced" fraction x is analogous to the momentum ordering $x\leq z$ in the DGLAP equation: the produced

fraction cannot be larger than the original one. In the present case, if the parton also takes some nonzero fraction α of the momentum transfer, the allowed values of x cannot exceed $z(1-\alpha^2)$.

For gluons, we combine the expansion for the bilocal operator

$$z^\mu z^\nu G_{\mu\alpha}(-z/2)G_{\alpha\nu}(z/2) = \sum_{n=0}^{\infty}(-1)^n\frac{3(2n+5)}{2^{2n+3}(n+2)!}\int_{-1}^{1}(1-\alpha^2)^{n+2}\mathcal{O}^G_{n+1}(\alpha z/2)\,d\alpha \tag{A10}$$

and the expression (A2) for the Gegenbauer moments at $\xi=0$. The resulting relation for the moments $\tilde{f}^G_n(\alpha;\mu)$ has the form identical to Eq.(A6) derived in the quark case.

APPENDIX B: EVOLUTION EQUATIONS

In Ref. [3], the kernels $W^{ab}_\zeta(x,z)$ were obtained *via*

$$W^{ab}_\zeta(x,z) = \int_0^1\int_0^1 B^{ab}(u,v)\delta\left(x-\bar{u}z+v(z-\zeta)\right)\theta(u+v\leq 1)\,du\,dv \tag{B1}$$

from the evolution kernels for the light-ray operators [18,22]:

$$B^{qq} = \frac{\alpha_s}{\pi}C_F\left(1+\delta(u)[\bar{v}/v]_+ + \delta(v)[\bar{u}/u]_+ - \frac{1}{2}\delta(u)\delta(v)\right) \tag{B2}$$

$$B^{gq} = \frac{\alpha_s}{\pi}C_F\left(2+\delta(u)\delta(v)\right) \tag{B3}$$

$$B^{qg} = \frac{\alpha_s}{\pi}N_f(1+4uv-u-v) \tag{B4}$$

$$B^{gg} = \frac{\alpha_s}{\pi}N_c\left(4(1+3uv-u-v)+\frac{\beta_0}{2N_c}\delta(u)\delta(v)+\delta(u)\bar{v}^2\left[\frac{1}{v}\right]_+ + \delta(v)\bar{u}^2\left[\frac{1}{u}\right]_+\right) \tag{B5}$$

In the nonsinglet case, only the qq kernel is needed. Using Eqs. (B2),(B1), one can easily derive the rules allowing to transform each of the four terms contained inside the bracket in Eq. (B2)

$$1 \to \theta(x<\zeta)\int_0^x \frac{\zeta-x}{\zeta-z}\mathcal{F}^q_\zeta(z)\frac{dz}{\zeta} + \int_x^1\left\{\theta(x<\zeta)\frac{x}{\zeta}\frac{dz}{z}+\theta(x>\zeta)\frac{z-x}{z-\zeta}\right\}\mathcal{F}^q_\zeta(z) \equiv d_1, \tag{B6}$$

$$\begin{aligned}\delta(u)\left[\frac{\bar{v}}{v}\right]_+ &\to \int_0^{\min\{x/\zeta,\bar{x}/\bar{\zeta}\}}\left[\mathcal{F}^q_\zeta\left(\frac{x-v\zeta}{\bar{v}}\right)-\mathcal{F}^q_\zeta(x)\right]\frac{dv}{v}+\left[1+\log(\min\{x/\zeta,\bar{x}/\bar{\zeta}\})\right]\mathcal{F}^q_\zeta(x)\\ &= \theta(x<\zeta)\left\{\int_0^{x/\zeta}\left[\mathcal{F}^q_\zeta\left(\frac{x-v\zeta}{\bar{v}}\right)-\mathcal{F}^q_\zeta(x)\right]\frac{dv}{v}+\left[1+\log(x/\zeta)\right]\mathcal{F}^q_\zeta(x)\right\}\\ &+\theta(x>\zeta)\left\{\int_0^{\bar{x}/\bar{\zeta}}\left[\mathcal{F}^q_\zeta\left(\frac{x-v\zeta}{\bar{v}}\right)-\mathcal{F}^q_\zeta(x)\right]\frac{dv}{v}+\left[1+\log(\bar{x}/\bar{\zeta})\right]\mathcal{F}^q_\zeta(x)\right\}\equiv d_2,\end{aligned} \tag{B7}$$

$$\delta(v)\left[\frac{\bar{u}}{u}\right]_+ \to \int_0^x\left[\mathcal{F}^q_\zeta(x/\bar{u})-\mathcal{F}^q_\zeta(x)\right]\frac{du}{u}+(1+\log(\bar{x}))\,\mathcal{F}^q_\zeta(x)\equiv d_3, \tag{B8}$$

$$-\frac{1}{2}\delta(u)\delta(v) \to -\frac{1}{2}\mathcal{F}^q_\zeta(x)\equiv d_4\,. \tag{B9}$$

In terms of these contributions, the total qq part of (6.1) is given by

$$\hat{W}^{qq}_{\zeta} \otimes \hat{\mathcal{F}}^{q}_{\zeta}(\mu) = \frac{\alpha_s}{\pi} C_F (d_1 + d_2 + d_3 + d_4). \tag{B10}$$

Expressions for d_1 and d_4 were obtained directly by performing integrations over u, v of the terms in Eq. (B2) with 1 and $(-\frac{1}{2}\delta(u)\delta(v))$. To obtain expressions for d_2 and d_3, we changed the order of integrations. For example, to obtain d_3, we first take the integral over v and then over z:

$$\int_0^1 \left(\int_0^1 \int_0^1 \delta(x - \bar{u}z + v(z-\zeta))\delta(v) \left[\frac{\bar{u}}{u}\right]_+ \mathcal{F}_\zeta(z)\, du\, dv \right) dz =$$
$$\int_0^1 \left(\int_0^1 \delta(x - \bar{u}z) \left[\frac{\bar{u}}{u}\right]_+ \mathcal{F}_\zeta(z)\, du \right) dz = \int_0^1 \frac{du}{\bar{u}} \theta(x/\bar{u} < 1) \left[\frac{\bar{u}}{u}\right]_+ \mathcal{F}_\zeta(x/\bar{u})$$
$$= \int_0^x \mathcal{F}_\zeta(x/\bar{u}) \frac{du}{u} - \mathcal{F}_\zeta(x) \int_0^1 \frac{\bar{u}}{u} du$$
$$= \int_0^x \frac{du}{u} \left[\mathcal{F}^q_\zeta(x/\bar{u}) - \mathcal{F}^q_\zeta(x) \right] + (1 + \log(\bar{x}))\, \mathcal{F}^q_\zeta(x). \tag{B11}$$

The resulting form for integrals in d_2 and d_3 is particularly convenient for numerical calculations, because the integrand in Eqs. (B7), (B8) is explicitly regular as $u, v \to 0$.

In a similar way, we represent the gg -kernel part as

$$\hat{W}^{gg}_{\zeta} \otimes \hat{\mathcal{F}}^{g}_{\zeta}(\mu^2) = \frac{\alpha_s}{\pi} N_c (d_1^g + d_2^g + d_3^g + d_4^g). \tag{B12}$$

where

$$d_1^g = 2\,\theta(x<\zeta) \left\{ \int_0^x \frac{\zeta - x}{\zeta^2(\zeta - z)^2} \left(\zeta(2 + x - z) - x^2 - z(2 - x) \right) \mathcal{F}^g_\zeta(z)\, dz + \int_x^1 \frac{x^2}{\zeta^2} \left(3 - \frac{2x}{\zeta} + \frac{\zeta - x}{z} \right) \mathcal{F}^g_\zeta(z) \frac{dz}{z} \right\}$$
$$+ 2\,\theta(x>\zeta) \int_x^1 \frac{z - x}{z - \zeta} \left(1 + \frac{x(x-\zeta)}{z(z-\zeta)} \right) \mathcal{F}^g_\zeta(z) \frac{dz}{z} \tag{B13}$$

$$d_2^g = \theta(x<\zeta) \left\{ \int_0^{x/\zeta} \left[\bar{v} \mathcal{F}^g_\zeta \left(\frac{x - v\zeta}{\bar{v}} \right) - \mathcal{F}^g_\zeta(x) \right] \frac{dv}{v} + \log(x/\zeta)\, \mathcal{F}^g_\zeta(x) \right\}$$
$$+ \theta(x>\zeta) \left\{ \int_0^{x/\bar{\zeta}} \left[\bar{z} \mathcal{F}^g_\zeta \left(\frac{x - v\zeta}{\bar{v}} \right) - \mathcal{F}^g_\zeta(x) \right] \frac{dv}{v} + \log(\bar{x}/\bar{\zeta}) \mathcal{F}^g_\zeta(x) \right\}, \tag{B14}$$

$$d_3^g = \int_0^x \left(\bar{u} \mathcal{F}^g_\zeta(x/\bar{u}) - \mathcal{F}^g_\zeta(x) \right) \frac{du}{u} + \log(\bar{x}) \mathcal{F}^g_\zeta(x), \tag{B15}$$

$$d_4^g = \frac{\beta_0}{2N_c} \mathcal{F}^g_\zeta(x) \tag{B16}$$

The qg and gq parts of the evolution equations (6.2) for untilded NFPDs in the region where $z < \zeta$ cannot be unambiguously reconstructed from the light-ray kernels. We use the form suggested in [23], which leads to

$$s^{gq} = \theta(x>\zeta) \frac{\zeta - x}{\zeta} \int_x^1 \frac{z - x}{z - \zeta} \left(1 + \frac{x(x-\zeta)}{z(z-\zeta)} \right) \mathcal{F}^q_\zeta(z) \frac{dz}{z} \tag{B17}$$

$$+ \theta(x<\zeta) \left\{ \frac{(\zeta - x)^2}{\zeta} \int_0^x \frac{1}{\zeta - z} \left(1 - \frac{2z}{\zeta} + 4\frac{x(\zeta - z)}{\zeta^2} \right) \mathcal{F}^q_\zeta(z)\, dz \right. \tag{B18}$$

$$\left. + \frac{x^2}{\zeta^2} \int_x^1 \left(-1 + \frac{2z}{\zeta} + 4\frac{z(\zeta - x)}{\zeta^2} \right) \mathcal{F}^q_\zeta(z)\, dz \right\}, \tag{B19}$$

$$s^{qg} = \theta(x>\zeta) \int_x^1 \frac{(z-x)^2 + x(x-\zeta)}{z^2(z-\zeta)^2} \left(1 + \frac{x(x-\zeta)}{z(z-\zeta)} \right) \mathcal{F}^q_\zeta(z)\, dz \tag{B20}$$

$$+ \theta(x<\zeta) \left\{ \frac{-(\zeta - x)}{\zeta} \int_0^x \frac{1}{(\zeta - z)^2} \left(1 - \frac{2x}{\zeta} - 4\frac{x(\zeta - z)}{\zeta^2} \right) \mathcal{F}^q_\zeta(z)\, dz \right. \tag{B21}$$

$$\left. + \frac{x}{\zeta} \int_x^1 \frac{1}{z^2} \left(-1 + \frac{2x}{\zeta} - 4\frac{z(\zeta - x)}{\zeta^2} \right) \mathcal{F}^q_\zeta(z)\, dz \right\}. \tag{B22}$$

In the matrix notation

$$\hat{W}^{qq}_{\zeta} \otimes \hat{\mathcal{F}}^{q}_{\zeta}(\mu^2) = \frac{\alpha_s}{\pi} C_f s^{qq}, \tag{B23}$$
$$\hat{W}^{qg}_{\zeta} \otimes \hat{\mathcal{F}}^{g}_{\zeta}(\mu^2) = \frac{\alpha_s}{\pi} N_f s^{qg}. \tag{B24}$$

[1] X. Ji, Phys.Rev.Lett. **78** 610 (1997); Phys.Rev. D **55** (1997) 7114.

[2] A.V. Radyushkin, Phys. Lett. B **380** (1996) 417; Phys. Lett. B **385** (1996) 333.

[3] A.V. Radyushkin, Phys. Rev. D **56** (1997) 5524.

[4] J.C. Collins, L. Frankfurt and M. Strikman, Phys. Rev. D **56** (1997) 2982.

[5] D. Müller, D. Robaschik, B. Geyer, F.-M. Dittes and J. Hořejši, Fortschr.Phys. **42** (1994) 101.

[6] A.V. Radyushkin, Phys. Rev. D **59** (1999) 014030.

[7] A.V. Radyushkin, Phys. Lett. B **449** (1999) 81.

[8] A.G. Shuvaev, K.J. Golec-Biernat, A.D. Martin and M.G. Ryskin, DTP-99-18 (1999), hep-ph/9902410.

[9] M.V. Polyakov and C. Weiss, RUB-TPII-1-99, hep-ph/9902451.

[10] A.V. Radyushkin, Phys. Lett. B **131** (1983) 179.

[11] A.V. Radyushkin, Phys. Rev. D **58** (1998) 114008.

[12] L. Mankiewicz, G. Piller and T. Weigl, Eur. Phys. J. C **5** (1998) 119.

[13] X.Ji, J. Phys. G **24** (1998) 1181.

[14] A.D. Martin and M.G. Ryskin, Phys.Rev.D **57** 6692 (1998).

[15] L. Frankfurt, A. Freund, V. Guzey and M. Strikman, Phys. Lett. B **418** 345 (1998); Erratum, ibid. B **429** 414 (1998); A. Freund and V. Guzey, hep-ph/9806267; hep-ph/9801388.

[16] A.V. Belitsky, B. Geyer, D. Muller and A. Schäfer, Phys. Lett. B **421** 312 (1998); A.V. Belitsky, D. Muller, L. Niedermeier and A. Schäfer, Phys. Lett. B **437** 160 (1998).

[17] K.J. Golec-Biernat, A.D. Martin and M.G. Ryskin, DTP-99-24 (1999), hep-ph/9903327.

[18] I.I. Balitsky and V.M.Braun, Nucl.Phys. B **311** (1988/89) 541.

[19] A.V. Efremov and A.V. Radyushkin, Theor. Math. Phys. **42** (1980) 97.

[20] D.J. Gross and F. Wilczek, Phys. Rev. D **9** (1974) 980.

[21] H. Georgi and H.D. Politzer, Phys. Rev. D **9** (1974) 416.

[22] Th. Braunschweig, B. Geyer and D. Robaschik, Ann. Phys. (Leipzig) **44** (1987) 403.

[23] A.V. Belitsky and D. Müller, Nucl.Phys. B **527** (1998) 207.