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The Southeastern Universities Research Association (SURA) operates the Thomas Jefferson National Accelerator Facility for the United States Department of Energy under contract DE-AC05-84ER40150

WM-98-104
JLAB-THY-98-35

Confinement through a relativistic generalization of the linear interaction*

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Abstract

Generalizing a covariant framework previously developed, it is shown that confinement insures that meson $\rightarrow q + \bar{q}$ decay amplitudes vanish when both quarks are on-shell. Regularization of singularities in a covariant linear potential associated with nonzero energy transfers (i.e. $q^2 = 0, q^\mu \neq 0$) is improved.

*Contributed paper to the Third International Conference On Quark Confinement, Jefferson Lab, Newport News, Virginia, June 7-12 1998

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I. INTRODUCTION

Even for a simple system such as a quark-antiquark (referred to collectively “quarks”) bound state, color confinement implies that the matrix element for meson decay, $\mu \rightarrow q + \bar{q}$, must vanish whenever this decay is kinematically possible. This trivial statement can be realized by two possible mechanisms. Either i) the quark propagators are free of timelike mass poles, as is usually assumed in Euclidean metric based studies [1], or ii) the vertex function of the bound state **vanishes** when both quarks are on-shell. In an earlier work [2] it was assumed that the quark propagators have mass poles, but it was not known that relativistic generalization of the linear confining potential guaranteed the correct vanishing of the vertex function. In this work it is shown that the vanishing of the vertex function is a general feature of any confining interaction, and that **insisting on the correct nonrelativistic limit leads naturally to the second option.**

II. THE NONRELATIVISTIC LINEAR POTENTIAL

It was shown earlier [2] that the nonrelativistic linear interaction

$$\mathcal{V}(r) = -C + \sigma r, \quad (2.1)$$

can be expressed in momentum space by

$$\mathcal{V}(\vec{q}) = \lim_{\epsilon \rightarrow 0} \left[V_A(\vec{q}) - \delta^3(\vec{q}) \int d^3 q' V_A(\vec{q}') \right] - (2\pi)^3 \delta^3(\vec{q}) C, \quad (2.2)$$

where

$$V_A(\vec{q}) \equiv -8\pi\sigma \frac{1}{(\vec{q}^2 + \epsilon^2)^2}. \quad (2.3)$$

in coordinate space

$$V_A(\vec{r}) = \int \frac{d^3 q}{(2\pi)^2} e^{-iq \cdot r} V_A(\vec{q}) = -\sigma \frac{e^{-\epsilon r}}{\epsilon} \propto \lim_{\epsilon \rightarrow 0} \sigma \left(r - \frac{1}{\epsilon} \right), \quad (2.4)$$

and hence the delta function subtraction cancels the infinite $1/\epsilon$ term, insuring that the limit $\epsilon \rightarrow 0$ exists and that the potential confines. It also insures that the linear potential vanishes at the origin. In fact, the delta function subtraction could be used to construct confining potentials for any monotonically increasing $V_A(r)$ for which $V_A(r) - V_A(0) = \infty$ for some r . For example, after the delta

function subtraction even the coulomb potential $V_A(r) = -1/r$ would result in a confining interaction.

Using the linear potential Eq. (2.2) the Schrödinger equation for two quarks of equal mass m becomes

$$\left[\frac{\vec{p}^2}{m} - E \right] \Psi(\vec{p}) = - \int \frac{d^3 k}{(2\pi)^3} V_A(\vec{p} - \vec{k}) \left[\Psi(\vec{k}) - \Psi(\vec{p}) \right] + C \Psi(\vec{p}), \quad (2.5)$$

where E is the binding energy.

III. THE RELATIVISTIC GENERALIZATION

A seemingly natural candidate for a relativistic generalization of the linear interaction (2.2) is to use the Bethe-Salpeter equation with a kernel given by

$$\mathcal{V}(\vec{q}) \xrightarrow{?} \lim_{\epsilon \rightarrow 0} \left[V_A(q) - \delta^4(q) \int d^4 q' V_A(q') \right] - (2\pi)^3 \delta^4(q) C, \quad (3.1)$$

where $\vec{q}^2 \rightarrow -q^2$ in Eq. (2.3). Unfortunately this form does not have the correct nonrelativistic limit. Therefore we rephrase the question: Can we find a covariant equation that reduces to the Schrödinger equation with linear interaction in the nonrelativistic limit? In order to motivate the relativistic equation, start by defining the Schrödinger *vertex* function $\Phi(\vec{p})$

$$\Phi(\vec{p}) \equiv \left[\frac{\vec{p}^2}{m} - E \right] \Psi(\vec{p}). \quad (3.2)$$

The Schrödinger equation with linear interaction is then

$$\Phi(\vec{p}) = - \int \frac{d^3 k}{(2\pi)^3} m V_A(\vec{p} - \vec{k}) \left[\frac{\Phi(\vec{k})}{(\vec{k}^2 - mE)} - \frac{\Phi(\vec{p})}{(\vec{p}^2 - mE)} \right] + mC \frac{\Phi(\vec{p})}{(\vec{p}^2 - mE)}. \quad (3.3)$$

Introducing the four-vectors P , k , and p , with $P = (2m + E, \vec{0})$ and $p^2 = k^2 = m^2$ (so that $E_k = \sqrt{m^2 + \vec{k}^2}$), the relativistic generalization of (3.3) which reduces to it in the nonrelativistic limit (when $m \rightarrow \infty$) is

$$\Phi(p) = - \int \frac{d^3 k}{(2\pi)^3} \frac{2m^2}{E_k} V_A(p - k) \left[\frac{\Phi(k)}{m^2 - (P - k)^2} - \frac{\Phi(p)}{m^2 - (P - p)^2} \right] + 2mC \frac{\Phi(p)}{m^2 - (P - p)^2}. \quad (3.4)$$

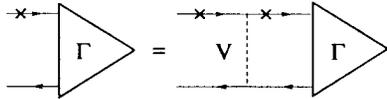


FIG. 1. The Gross equation. The \times indicate that the particle is on-shell.

is the Gross equation (see Fig. 1) and the discussion shows that this is the most natural relativistic equation obtained from a generalization of the nonrelativistic Schrödinger equation for confined particles.

In a previous application of the Gross equation [2] the kernel (2.3), with $\rightarrow -q^2$, was used. This choice is undesirable for use with the two channel version of this equation, where the mass shell constraints which fix q_0 introduce singularities for non-zero momentum transfers (i.e. $q^2 = 0$, but $q^\mu \neq 0$). To rectify this problem the kernel $V_A(q)$ is defined to be

$$V_A(q) \equiv -8\pi\sigma \frac{1}{q^4 + (P \cdot q)^4 / P^4} \quad (3.5)$$

advantages of this form are many: **i)** singularities are restricted to $q^\mu = 0$, **ii)** interaction strength *does not* depend on the bound state momentum P in the rest frame, **iii)** it has the correct nonrelativistic dependence on q^2 , and **iv)** the infrared regularization used previously [2] is no longer needed.

IV. PROOF OF CONFINEMENT

It is sufficient to consider the case when the constant term $C = 0$. If the mass of the bound state $\mu > 2m$, then there exists a value of three momentum $|\vec{p}| = p_c$ when both quarks can be on-shell. In this case $(P - p)^2 = \mu^2 - 2\mu E_{p_c} + m^2 = m^2$, and the subtraction term in Eq. (3.4) appears to be singular. This singularity is cancelled by the first term, and hence, if the equation is to have a solution, the vertex function must be zero at p_c . In particular, as $|\vec{p}| \rightarrow p_c \pm \epsilon$,

$$(P - p)^2 - m^2 \rightarrow \mp \frac{2\mu p_c \epsilon}{E_{p_c}}, \quad (4.1)$$

if the vertex function would have an infinite discontinuity at p_c unless it were zero there. We conclude that $\Phi(p) = 0$, when $2E_p = \mu$ (as illustrated in Fig. 2).

A numerical confirmation of this result is shown in Fig. 3. In the actual calculations to be presented in an upcoming paper, the bound state equation is averaged over positive and negative energy contributions and symmetrized by taking up the pole contributions of both constituents. The average over positive

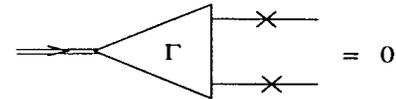


FIG. 2. When both constituents are on-shell, the vertex function vanishes.

and negative energy contributions leads to a two channel equation with a two component vertex function. In Fig. 3 the solid line is the (large) component in which the off-shell quark has positive energy and the dashed line is the (small) component in which the off-shell quark has negative energy. Since a physical decay must produce two positive energy quarks, only the large component must have the “confinement” node. Fig. 3 shows the vertex functions for an excited pion with mass $\mu = 1.2$ GeV composed of two quarks with masses $m = 0.34$ MeV. The large vertex function must have *two* nodes: one due to the excitation and another at $p^2 = 0.244$ GeV², exactly where both particles are on-shell.

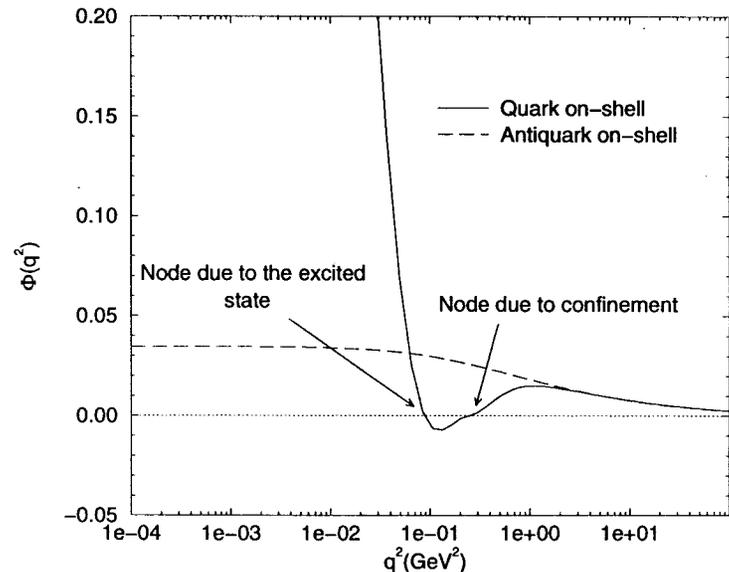


FIG. 3. The pion 1st excited state Gross vertex functions are shown. The first node is due to the excited state. The second node assures that the bound state does not decay.

V. CONCLUSIONS

The relativistic equation obtained from a generalization of the nonrelativistic Schrödinger equation for confined particles is of the Gross equation type rather than the Bethe-Salpeter equation type. The confinement mechanism arises not from the lack of quark mass poles, but through the vanishing of the vertex function when both quarks are on their positive energy mass-shell. Because of this confinement mechanism, the vertex functions (ground or excited state) have one additional node if the bound state is heavy enough.

ACKNOWLEDGEMENT

This work was supported in part by the US Department of Energy under Grant No. DE-FG02-97ER41032.

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