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Massless Fermions on the Lattice.

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Abstract

We consider a nonlocal lattice action for fermions fermion doubling in lattice theories.
It is shown, that it is possible to avoid the fermionic doubling in the case of free
fermions, but this approach does not reproduce results for the effective action for gauge
fields in the continuum theory, because the high frequency fermion modes have a strong
dependence on the gauge field.

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I. INTRODUCTION

Nowadays, lattice techniques have become an important method for the study of non-perturbative quantum field theory. It is widely used for calculations of hadron spectrum and for understanding the structure of nonperturbative effects in quantum field theory [1]. However, there is a serious problem to avoid the fermion doubling and to implement the axial anomaly into the lattice field theory. The most popular way to resolve the problem was suggested by Wilson [2] and in ref. [3] it was shown, that in the continuum limit, the lattice theory with Wilson fermions reproduces axial anomaly. The Wilson action contains a term which breaks chiral invariance by mass-like term, with a mass parameter of the order of inverse size of a lattice size and, due to interactions, this term is renormalized and fine tuning is then required. The Kogut-Susskind action [4] reduces the number of fermions but does not reproduce the axial anomaly. Also, there are attempts to resolve the problem by a gauge-violating Majorana-type Wilson mass [5].

Recently, there has been a considerable theoretical activity on the use of surface states as a basis for a theory of chiral lattice fermions [6] where our four dimensional world is considered as an interface in a five dimensional underlying space [7]. Under some conditions, low energy fermionic states are bound to this wall. For this low energy states, we have an effective chiral theory on the interface. A Hamiltonian approach for this formulation of the chiral theory was considered in ref. [8].

In this paper, we study nonlocal gauge invariant action for massless fermions without fermion doubling.

II. THE DOUBLING PROBLEM AND NONLOCAL LATTICE ACTION FOR FERMIONS.

Let us consider a general one-dimensional lattice action for a free massless fermion field:

$$S = \sum_{n,m} \bar{\psi}_m A_{m,n} \psi_n \quad (1)$$

where

$$A_{m,n} = -A_{n,m} = A_{m-n} \quad A(0) = 0 \quad (2)$$

In this paper, we consider the case of antiperiodic boundary conditions, which are imposed on the fermion field for convenience only. The case of periodical

boundary conditions could be considered as well; nothing would be changed except minor technical details.

Let us consider the lattice with even numbers of elements,

$$-N \leq n \leq N-1, \quad (3)$$

we conveniently rewrite the action (1) in the following form

$$S = \sum_{n,m} \bar{\psi}_{m+\frac{1}{2}} A_{m-n} \psi_{n+\frac{1}{2}} \quad (4)$$

where $2N$ is the size of the lattice.

To diagonalize the action (1), we use Fourier transformation:

$$\begin{aligned} \psi_{m+\frac{1}{2}} &= \sqrt{\frac{1}{2N}} \sum_{k=-N}^{N-1} e^{i\frac{\pi(k+\frac{1}{2})(m+\frac{1}{2})}{N}} \Psi_{k+\frac{1}{2}} \\ \bar{\psi}_{m+\frac{1}{2}} &= \sqrt{\frac{1}{2N}} \sum_{k=-N}^{N+1} e^{-i\frac{\pi(k+\frac{1}{2})(m+\frac{1}{2})}{N}} \bar{\Psi}_{k+\frac{1}{2}} \end{aligned} \quad (5)$$

In the new variables, the action (1) has the form:

$$S = \frac{1}{2N} \sum_{m,n,k,l} \bar{\Psi}_{k+\frac{1}{2}} e^{-i\frac{\pi(k+\frac{1}{2})(m+\frac{1}{2})}{N}} A(m-n) e^{i\frac{\pi(l+\frac{1}{2})(m+\frac{1}{2})}{N}} \Psi_{l+\frac{1}{2}} \quad (6)$$

After summation over m for fixed value of $(m-n)$, we obtain:

$$\begin{aligned} S^{fer.} &= \sum_{k,n=-N}^{N-1} \bar{\Psi}_{k+\frac{1}{2}} e^{-i\frac{\pi(k+\frac{1}{2})(n+\frac{1}{2})}{N}} A(n) \Psi_{k+\frac{1}{2}} \\ &= -i \sum_{k=-N}^{N-1} \bar{\Psi}_{k+\frac{1}{2}} \left(2 \sum_{n=0}^{N-1} A(n) \sin\left(\frac{\pi(k+\frac{1}{2})n}{N}\right) \right) \Psi_{k+\frac{1}{2}} \\ &= \sum_{k=-N}^{N+1} \bar{\Psi}_{k+\frac{1}{2}} B\left(k+\frac{1}{2}\right) \Psi_{k+\frac{1}{2}} \end{aligned} \quad (7)$$

where

$$B(k) = -i2 \sum_{n=0}^{N-1} A(n) \sin\left(\frac{\pi(k+\frac{1}{2})n}{N}\right). \quad (8)$$

In the case of the standard lattice action for fermions

$$\begin{aligned} A(n) &= i \frac{\delta^{n1}}{2a}, \\ \omega_{k+\frac{1}{2}} &= \frac{\pi(k+\frac{1}{2})}{Na}, \end{aligned} \quad (9)$$

where a is the lattice spacing and ω is the fermion momentum. Then the action has the well known form:

$$S = \sum_k \bar{\Psi}_{k+\frac{1}{2}} \frac{\sin(\omega_{k+\frac{1}{2}} a)}{a} \Psi_{k+\frac{1}{2}} \quad (10)$$

and for $\omega_{N-\frac{1}{2}} = \frac{\pi(N-\frac{1}{2})}{Na}$, in the limit $N \rightarrow \infty$, we obtain an additional pole in the fermion propagator.

Now, let us consider $B(k)$ for $k = N - \frac{1}{2}$ in the general nonlocal case:

$$\begin{aligned} B(N - \frac{1}{2}) &= 2 \sum_{n=0}^{N-1} A(n) \sin\left(\frac{\pi(N-\frac{1}{2})n}{N}\right) \\ &= 2 \sum_{n=0}^{N-1} A(n) \sin\left(\pi n - \frac{\pi n}{2N}\right) \\ &= 2 \sum_{n=0}^{N-1} A(n) (-1)^{n+1} \sin\left(\frac{\pi n}{2N}\right) \end{aligned} \quad (11)$$

It is clear, that if we choose that as $n \rightarrow \infty$

$$A(n) \sim \frac{(-1)^{n+1}}{n}, \quad (12)$$

the second pole in the fermion propagator will be absent in the continuum limit.

It is convenient to choose $A(n)$ in the following form

$$A(m-n) = \sum_{\alpha=-N}^{N-1} \langle E_{m+\frac{1}{2}} e_{\alpha+\frac{1}{2}} \rangle \omega_{\alpha+\frac{1}{2}} \langle e_{-(\alpha+\frac{1}{2})} E_{n+\frac{1}{2}} \rangle \quad (13)$$

where we use the following notations:

$$e_{\alpha+\frac{1}{2}}(x) = \sqrt{\frac{1}{L}} e^{i\omega_{\alpha+\frac{1}{2}} x}$$

$$\begin{aligned} E_{m+\frac{1}{2}}(x) &= \sqrt{\frac{1}{a}} \Theta(x-ma) \Theta((m+1)a-x) \\ \langle f \rangle &= \int_{-L/2}^{L/2} f(x) dx \end{aligned} \quad (14)$$

Then, in the limit $N \rightarrow \infty$, we obtain:

$$A(n) = i \frac{1}{a} \int_0^a \frac{dx}{\pi} \sin(xn) \frac{2(1-\cos(x))}{x} \quad (15)$$

$$B(k + \frac{1}{2}) = \omega_{k+\frac{1}{2}} \left(\frac{2(1-\cos(\omega_{k+\frac{1}{2}} a))}{\omega_{k+\frac{1}{2}}^2 a^2} \right) \quad (16)$$

From eq.(16), we see that the second pole of the fermion propagator is absent. Notice, that asymptotically $A(n)$ in eq.(15) has the form (12).

Also, one may use the simplest choice for $A(m-n)$:

$$A(n) = \frac{(-1)^{n+1}}{an} \quad n \neq 0 \quad (17)$$

In this case, the second pole in the propagator of a free fermion will be absent in the continuum limit as well. The case $A(n) \sim (-1)^n$ was considered in [10].

Thus, we have constructed the action for a free massless fermion field with a single fermion. Still there is a question: does the theory have a correct limit at $N \rightarrow \infty$ in the presence of gauge fields.

To study this question, let us consider the lattice fermion in the presence of an external gauge field. In the case of $D=1$, $U(1)$ gauge theory with one fermion, the action has the following form:

$$S = \sum_{m,n=-N}^{N-1} \bar{\psi}_{m+\frac{1}{2}} V^{(m+\frac{1}{2})(n+\frac{1}{2})} A(m-n) \psi_{n+\frac{1}{2}}, \quad (18)$$

where we introduce the following matrices on the links of the lattice:

$$U^{m+\frac{1}{2}} = \mathcal{P} \exp \left(ig \int_{x=(m-\frac{1}{2})a}^{x=(m+\frac{1}{2})a} A(x) dx \right) \quad (19)$$

and

$$\begin{aligned} V^{(m+\frac{1}{2})(n+\frac{1}{2})} &= U^{m+1/2} U^{m-3/2} \dots U^{n+1/2} \\ \text{for } 0 < |m-n|_{\text{mod}(2N)} < N \end{aligned}$$

$$V^{(m+\frac{1}{2})(n+\frac{1}{2})} = U^\dagger(m-1/2)U^\dagger(m+1/2)\dots U^\dagger(n-1/2)$$

$$\text{for } -N < |m-n|_{\text{mod}(2N)} < 0 \quad (20)$$

\mathcal{P} denotes a path-ordered product,

$$|m|_{\text{mod}(2N)} = m - (2N)j \quad \text{at } -N \leq (m - (2N)j) \leq N - 1$$

$$j = 0, \pm 1, \pm 2, \dots \quad (21)$$

Here and below we impose periodic boundary conditions for the gauge field \mathcal{A} . It is known, that we can gauge away contributions of nonconstant components of the gauge field \mathcal{A}^n , where $\mathcal{A}(x) = \sum_n \mathcal{A}^n e^{i\omega_n x}$. But in general case, it is not possible to remove a constant gauge field because there are gauge transformations which violate the boundary conditions:

$$\mathcal{A} \rightarrow \mathcal{A} - \frac{1}{g} \partial_x \alpha(x)$$

$$\alpha(x) = g \mathcal{A} x$$

$$\psi_{n+1/2} \rightarrow e^{i\alpha(x=a(n+1/2))} \psi_{n+1/2} \quad (22)$$

and keeps antiperiodic boundary conditions only when

$$\alpha(x) = (2\pi/L)nx, \quad n = \pm 1, \pm 2, \dots \quad (23)$$

The presence of the constant gauge field corresponds to the shift of momentum in eq.(8):

$$B(k+1/2) \rightarrow B(k+1/2+y/2) \quad (24)$$

where $y = \frac{gAL}{\pi}$.

Notice that the highest fermionic modes ($y \sim N$) have a strong dependence on the gauge field. It means that the contributions of the high frequency modes in physical observables (which feel the lattice structure) will not die in the limit $N \rightarrow \infty$.

Let us check this statement in the simplest case of the 2-dimensional Schwinger model with one Dirac fermion. In the case of continuous imaginary time and a lattice in space, the action has the following form:

$$S = \int dx \sum_{m,n=-N}^{N-1} (\bar{\psi}_{m+1/2}(x) iD_x \gamma_1 \delta^{mn} \psi_{n+1/2}(x)$$

$$+ \bar{\psi}_{m+1/2}(x) V^{(m+\frac{1}{2})(n+\frac{1}{2})} A(m-n) \psi_{n+1/2}(x) \quad (25)$$

where D_x is the covariant derivative:

$$iD_x \psi_{n+1/2}(x) = (i\partial_x - gA_x^{n+1/2}(x)) \psi_{n+1/2}(x) \quad (26)$$

It is easy to calculate the effective action for a constant gauge field \mathcal{A} . According to the continuous theory, the potential has the following form:

$$W(y) - W(0) = N_f \frac{\pi}{2L^2} (|y|_{\text{mod}(2)})^2 \quad (27)$$

where $|y|_{\text{mod}(2)} = y + 2j$ if $|y + 2j| < 1$ and $j = 0, \pm 1, \pm 2, \dots$, N_f is the number of Dirac fermions. In the case of the lattice theory we have:

$$W(y) - W(0) = -\frac{1}{2L} \int_{-\infty}^{\infty} d\omega \sum_{k=-N}^{N-1} \ln \left(\frac{\omega^2 + B^2(k+1/2+y/2)}{\omega^2 + B^2(k+1/2)} \right)$$

$$= -\frac{1}{2L^2} \sum_{k=-N}^{N-1} (|B(k+1/2+y/2)| - B(k+1/2)) \quad (28)$$

We considered the cases of standard (9) and nonlocal (17) fermion actions. The results obtained for $N = 10$ are shown in Fig.1, where solid lines corresponds to the standard lattice action with one Dirac fermion and to eq.(27) for $N_f = 2$ (two curves practically coincide with each other), and the dashed line corresponds to the case of the nonlocal lattice action for one Dirac fermion field. Comparing our results with eq.(27), we see that the standard fermionic action corresponds to the case of two Dirac fermions in the continuum limit. At the same time we see that the lattice with nonlocal action for fermions has a strong dependence on the lattice size and does not tend to any continuous theory with a finite number of fermions. Thus, we see that the lattice effects do not disappear in the case of nonlocal action. It is possible, of course, to discuss a possibility to kill this strong dependence by choosing a special form for $A(m)$, but in this case we need to make a fine tuning and it is not clear, whether these lattice artifacts disappear in the other physical values in the limit of $N \rightarrow \infty$.

Thus, we have to conclude that the attempt to resolve the fermion doubling problem by means of nonlocal action in the case of gauge theory has serious difficulties to obtain a correct continuum limit.

III. DISCUSSIONS

In this paper, we considered nonlocal fermion action to solve the problem of the fermion doubling.

It was found that it is possible to construct a nonlocal lattice action for free fermions without doubling, but the theory has no correct continuum limit in the presence of a gauge field. This problem appears because of the strong couplings of high frequency modes which feel lattice structure.

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REFERENCES

- [1] M. Creutz, "Quarks, gluons and lattices", Cambridge University Press 1983.
- [2] K. G. Wilson, Phys. Rev. D10 (1974) 2445.
- [3] A. Coste, C. Korthals-Altes, O. Napoly Phys. Lett. B179 (1986) 125; Nucl. Phys. B289 (1987) 645.
- [4] J. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395.
- [5] C. Pryor, Phys. Rev. D43 (2669).
- [6] D. Kaplan, Phys. Lett. B288 (1992) 342; M. Golterman, K. Jansen, D. Kaplan, Phys. Lett. B301 (1993) 219.
- [7] C.P. Korthals-Altes, S. Nicolis, J. Prades, Phys. Lett. B316 (1993) 339.
- [8] M. Creutz and I. Horváth, preprint BNL-60062.
- [9] H. Nielsen and M. Ninomiya, Nucl. Phys. B185 (1981) 20; B193 (1981) 173.
- [10] S. V. Zenkin, Mod. Phys. Lett. A6 (1991) 151.
- [11] M.A. Shifman, Phys. Rep. 209 (1991) 341.

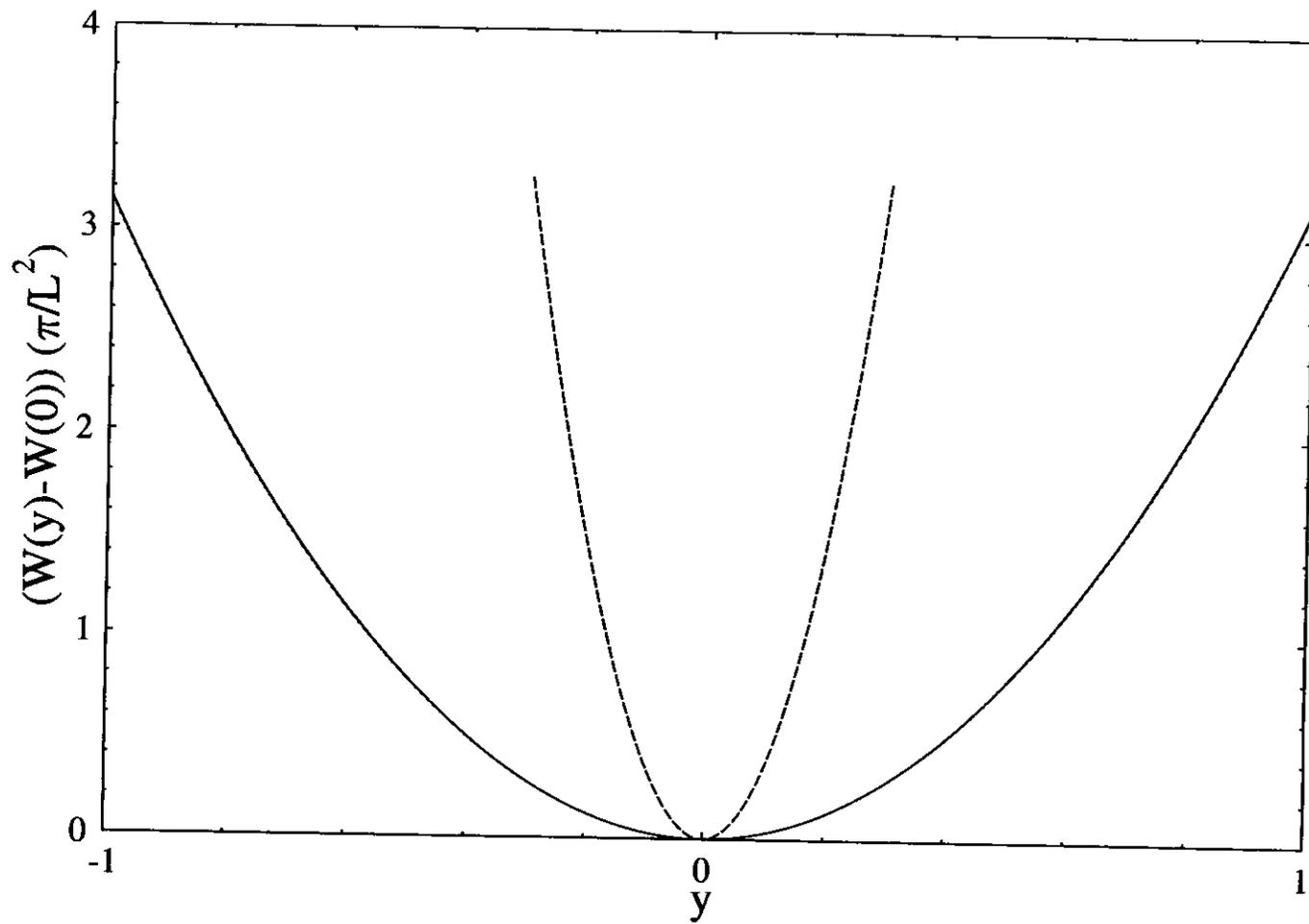


Fig.1.