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Lattice and Continuum Theories.

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Abstract

We investigate path integral formalism for continuum theory. It is shown that the path integral for the soft modes can be represented in the form of a lattice theory. Kinetic term of this lattice theory has a standard form and potential term has additional non-local terms which contributions should tend to zero in the limit of continuum theory. Contributions of these terms can be estimated. It is noted that this representation of path integral may be useful to improve lattice calculations taking into account hard modes contribution by standard perturbative expansion. We discuss translation invariance of correlators and the possibility to construct a lattice theory which keeps rotary invariance also.

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Path integral formalism [1] is one of the most useful tools to study a quantum field theory. However there is a serious problem to go out of boundaries of a perturbative theory. There are instanton calculations [2], a lattice calculation method [3] and variational approach which can be used in the case of quantum field theory [4] and sometimes it is possible to find nonperturbative exact results using symmetries of a quantum field model [5]. In ref. [6] it was used a cluster expansion to take into consideration nonperturbative effects.

In [7] it was studied lattice actions which give cut-off independent physical predictions even on coarse grained lattice. It was suggested to use *perfect* lattice action which is completely free of lattice artifacts. It was shown that in asymptotical free theories a combination of analytical and numerical techniques allows to find the perfect action to a sufficient precision.

Here we consider another possibility to improve lattice theory. In [8] it was proposed an alternative method for nonperturbative path integral computations. All modes are decomposed into hard (with $\omega^2 > \omega_0^2$) and soft (with $\omega^2 < \omega_0^2$) modes where ω_0 is a some parameter. It is clear that when a frequency is enough large then we can consider a potential term as a perturbation and use a conventional perturbative theory. Thus we can find an effective Lagrangian [9] for soft modes using wellknown perturbative theory. Soft modes contribution was estimated by strong coupling expansion. In [8] it was shown that this approach is applicable in the case of quantum mechanics with $V(x) = \lambda x^4$.

Here we show that the path integral for soft modes almost coincides with a standard lattice definition. To see that let us consider a path integral for quantum mechanics:

$$< x_f \mid e^{-\hat{H}t_0} \mid x_i >= \mathcal{N}^{-1} \int \mathcal{D}x(t) e^{-\int_0^{t_0} \mathcal{L}(x(t))dt} \tag{1}$$

where $\mathcal{L}(x(t)) = \frac{1}{2}(\frac{dx}{dt})^2 + V(x)$, $x(0) = x_i$, $x(t_0) = x_f$, $\hat{H}$ is a hamiltonian of a system, $\mathcal{N}$ is a normalization factor.

In the limit $t_0 \to \infty$ and with periodical boundary conditions $x_i(0) = x_f(t_0)$, we have

$$\mathcal{Z} = \int dx < x \mid e^{-\hat{H}t_0} \mid x >= \int dx < x \mid n > e^{-\varepsilon_n t_0} < n \mid x >_{|t_0 \to \infty} \tag{2}$$

$$= \int dx \mid \Psi_0(x) \mid^2 e^{-\varepsilon_0 t_0} = e^{-\varepsilon_0 t_0}$$

where ε_n is the energy of the n–th state, and ε_0 is the lowest energy of the system. The factor $\mathcal{N}$ is chosen in the following form: $\int \mathcal{D}x(t) e^{-\int_0^{t_0} \frac{1}{2}(\frac{dx}{dt})^2 dt}$.

In a perturbative theory the following basis for trajectories is used

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e_n(t) \tag{3}$$

where $e_n(t) = \frac{1}{\sqrt{t_0}} e^{i\omega_n t}$, $\omega_n = \frac{2\pi}{t_0} n$, $C_n = C^*_{-n}$.

This basis $\{e_n\}$ has the normalization: $< e_n \mid e_m >=< e_n^* e_m >= \int_0^{t_0} e_n^*(t) e_m(t) dt = \delta_{mn}$ and in the basis (3) path integral has the following form

$$\mathcal{Z} = \mathcal{N}^{-1} \int \prod_{n=-\infty}^{+\infty} \frac{dC_n}{\sqrt{2\pi}} e^{-<\mathcal{L}(\sum_n C_n e_n)>} \tag{4}$$

Here we use the denotation: $< f(t) >= \int_0^{t_0} f(t) dt$.

Hard modes can be taken into consideration by conventional perturbative procedure and after integration over them we obtain a low energy effective Lagrangian for the soft modes.

In [8] another basis for trajectories was suggested:

$$x(t) = \sum_{|n|<N} B_n E_n(t) + \sum_{|n|>N} C_n e_n(t) \tag{5}$$

$$\omega_0 = \frac{2\pi}{t_0} N;$$

Where $E_n(t) = \frac{1}{\sqrt{\Delta t}} \Theta(t - t_0/2 - n\Delta t)\Theta(t_0/2 + (n+1)\Delta t - t)$ and $\Delta t = \pi/\omega_0$.

The following denotations are used below: greek letters: μ, ν,..$= 0, \pm 1, .., \pm N$; small letters: m, n,..$= \pm(N+1), \pm(N+2), ..$; large letters: M, L,..$= 0, \pm 1, .. \infty$

In [8] it was shown that

$$\mathcal{Z} = \mathcal{N}^{-1} \int \prod_n \frac{dC_n}{\sqrt{2\pi}} \prod_\mu \frac{dB_\mu}{\sqrt{2\pi}} e^{-<\mathcal{L}(\sum(C_n e_n + B_\mu(E_\mu - <E_\mu e_n^*> e_n)))>} \mid J \mid \tag{6}$$

where $J = det(< e_\mu E_\nu >) = e^{-\frac{\omega_0 t_0}{\pi} j}$ where $j = \ln(\pi) - 1 \simeq 0.14$. In quantum mechanics lagrangian has a form

$$\mathcal{L} = \frac{1}{2}(\frac{dx}{dt})^2 + V(x) \tag{7}$$

Then in the terms of our basis $\{E_\mu\} + \{e_n\}$ the path integral is

$$\mathcal{Z} = \frac{1}{\mathcal{N}} \int \prod_n \frac{dC_n}{\sqrt{2\pi}} \prod_\mu \frac{dB_\mu}{\sqrt{2\pi}} |J| \tag{8}$$

$$\times \exp\{-[\frac{1}{2}|C_n|^2\omega_n^2 + \frac{1}{2}B_\mu < E_\mu e_\rho > \omega_\rho^2 < e_\rho^* E_\nu > B_\nu$$

$$+ < V(B_\mu(E_\mu - < E_\mu e_n > e_n) > +(terms\ with\ C_n)]\}$$

It is easy to show that the kinetic term for soft modes coincides with lattice definition: $\frac{1}{2}\frac{(x_{\mu+1}-x_\mu)^2}{\Delta}$ where $\Delta = \frac{\pi}{\omega_0}$ is the lattice size.

Potential for soft modes B_μ is

$$< V(B_\mu(E_\mu - < E_\mu e_n^* > e_n) > \tag{9}$$

$$=< V(B_\mu E_\mu) > - < (\frac{d}{dx}V(x)_{|x=B_\mu E_\mu} e_n >< e_n^* E_\nu > B_\nu + ...$$

Due to the definition $E_\nu(t)$ the first term in the expansion of eq.(9) coincides with lattice definition. The rest terms of the expansion in (9) are not presented in the lattice formulation of the path integral. So, suggesting that lattice theory and continuum formulation of the path integral describe the same system in the limit $\delta \to 0$, we have to conclude that in this limit ($\omega_0 \to \infty$) the contribution of the highest terms of expansion (9) should tend to zero and at a finite ω_0 we can estimate them. In [8] it was shown that correction of this nonlocal terms is about few percent at $\omega_0 \sim \lambda^{1/3}$ in quantum mechanics with $V(x) = \lambda x^4$.

Let us consider the soft modes contribution to correlator $< x(t_1), x(t_2) >$:

$$< x(t_1), x(t_2) >= \tag{10}$$

$$=< B_\mu(E_\mu(t_1) - < E_\mu e_m^* > e_m(t_1)), B_\nu(E_\nu(t_2) - < E_\nu e_n^* > e_n(t_2)) >$$

$$= e_\mu(t_1) < e_\mu^* E_\rho >< B_\rho B_\lambda >< E_\lambda e_\nu^* > e_\nu(t_2) = e_\mu(t_1) W_{\mu\nu} e_\nu(t_2)$$

Using that $< B_\rho B_\lambda >$ is a periodical function over ρ and λ it is possible to show that $W_{\mu\nu} \sim \delta_{\mu,-\nu}$. Thus we see that the correlator depends on $(t_1 - t_2)$ only. We see that translational invariance is not broken if we use any approximation of the path integral keeping periodical boundary conditions. It is true in the case of any correlators and it is possible to use this definition of correlators (10) in lattice calculations.

The formalism suggested here can be expanded to the case of $d-$dimensional scalar field theory directly using cubic lattice and the following definition for soft modes: $| p_i |< \omega_0$, p_i is one of components of particle momentum. The main features of the approach are: 1. Soft modes contribution can be calculated by lattice computations with additional nonlocal terms which should tend to zero at $\omega_0 \to \infty$; 2. It is possible to improve lattice results using effective lagrangian for soft modes; 3. All renormalizations are taking into account in effective lagrangian by a standard way; 4. Translational invariance is not broken in any approximation keeping periodical boundary conditions. Notice that rotary symmetry of quantum field theory is broken in this approach. It should be restored due to the hard modes contribution. It is possible to construct lattice theory which keeps rotary invariance. To obtain this kind of lattice theory it is enough to use cubic lattice for soft modes and rotational invariant basis for hard ones. It is possible to see that in this case kinetic term does not coincide with standard lattice term and it would be interesting to compare this kinetic term with the perfect action of [7]. The case of gauge theory should be investigated separately to try to find the way keeping gauge invariance.

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