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Reduced phase space quantization of Ashtekar's gravity on de Sitter background

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Abstract

We solve perturbative constraints and eliminate gauge freedom for Ashtekar's gravity on de Sitter background. We show that the reduced phase space consists of transverse, traceless, symmetric fluctuations of the triad and of transverse, traceless, symmetric fluctuations of the connection. A part of gauge freedom corresponding to the conformal Killing vectors of the three-manifold can be fixed only by imposing conditions on Lagrange multiplier. The reduced phase space is equivalent to that of ADM gravity on the same background.

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It is well known that quantum gravity is perturbatively non-renormalizable. Certain hopes that it will be possible to obtain a non-perturbative description of gravity are related to Ashtekar's variables [1], [2] in which the constraints become polynomial. The introducing of cosmological term in this formalism was considered in *Ref.3*. Several papers were devoted to the construction of the reduced phase space quantization in a framework of Ashtekar's formulation. We mention recent ones [4]. Certain progress was also achieved in the quantization of 2 + 1 dimensional models (see for example *Ref.5* and references therein). And there are many other related fields of research, as loop variables and minisuperspace models, which are not addressed here.

Apart from the non-renormalizability there are such internal problems arising in a framework of perturbative quantum gravity on de Sitter background as the lack of uniqueness of one-loop predictions which originates from ambiguous treatment of the zero-mode structure and non-covariance of the path integral measure. Recently this difficulty was resolved [6] in a framework of the ADM gravity [7]. It is interesting, however, to have a look into this problem from the point of view of another canonical approach.

In this paper we analyse linearized constraints of Ashtekar's gravity on de Sitter background and demonstrate that the complex reduced phase space [8] consists of symmetric, traceless, transverse fluctuations of the densitized triad and transverse, traceless, symmetric fluctuations of the connection. Transfer to the real phase space is done by imposing reality conditions. It is demonstrated that a part of gauge freedom can be fixed only by means of some additional conditions on Lagrange multiplier. This is essential for the path integral quantization since this results in an appearance of an additional Jacobian factor [6]. After this complete fixation of gauge freedom the real reduced phase space is proved to be equivalent to that of the ADM gravity. Our results generalize the results for flat background space reported in [2].

This work can be considered as a starting point for quantization of full non-linear theory in the case of non-trivial topology of space-time.

We begin with complex gravitational action in 3 + 1 dimensions

$$S = \int d^4x (iE_a^i \partial_i A_i^a - N^a G_a - N^i G_i - N G_0) \quad (1)$$

where, as usual, the densitized triad E_a^i and connection A_i^a are the canonical variables; G_a, G_i and G_0 are the Gauss law, the vector and the scalar constraints respectively

$$G_a = \mathcal{D}_i E_a^i = 0 \quad (2)$$

$$G_i = F_{ij}^a E_a^j = 0 \quad (3)$$

$$G_0 = \varepsilon^{abc} E_a^i E_b^j F_{ij}^c - \frac{\Lambda}{3} \varepsilon^{abc} \varepsilon_{ijk} E_a^i E_b^j E_c^k = 0, \quad (4)$$

Λ is the cosmological constants; N^a, N^i and N are Lagrange multipliers, \mathcal{D}_i is the covariant derivative with respect to the connection A_i^a , F_{ij}^a is the field strength. We use $i, j, k, l \dots$ to denote world indices, while $a, b, c, d \dots$ are reserved for Lorentz indices.

The reality conditions have the following form

$$E = E^*, \quad A + A^* = 2\Gamma(E), \quad (5)$$

where $\Gamma(E)$ is ordinary connection expressed in terms of E . There is another polynomial form [3] of the reality conditions. Perturbatively these two forms are equivalent because both of them ensure real evolution of the real triad.

Let us choose as a classical background the de Sitter space-time with the metric

$$dS^2 = -dt^2 + ch^2(t) d^2\Omega. \quad (6)$$

$d^2\Omega$ is the metric of unit three-sphere. For the sake of simplicity we put the overall scale factor in (6) to be equal to one. This corresponds to the cosmological constant $\Lambda = 3$.

To find perturbative reduced phase space of the theory, let us decompose canonical variables into background parts E and A , corresponding to the metric (6), and fluctuations H and B ,

$$E \rightarrow E + H, \quad A \rightarrow A + B.$$

The linearized constraints take the form

$$G_a = \mathcal{D}_i H_a^i + \varepsilon_{abc} B_i^b E_c^i = 0 \quad (7)$$

$$G_i = G_{ij}^a E_a^j + F_{ij}^a H_a^j = 0 \quad (8)$$

$$G_0 = 2\varepsilon^{abc} H_a^i E_b^j F_{ij}^c + \varepsilon^{abc} E_a^i E_b^j G_{ij}^c - \Lambda \varepsilon^{abc} \varepsilon_{ijk} H_a^i E_b^j E_c^k = 0 \quad (9)$$

\mathcal{D}_i is the background covariant derivative, F_{ij}^a is the background field strength and $G_{ij}^a = \partial_{[i} B_{j]}^a + \varepsilon^{abc} A_{[i}^b B_{j]}^c$.

We also need linearized gauge transformations of the fluctuations H and B . In general, an action of infinitesimal gauge transformation generated by a constraint G on a fluctuation of the variable Z reads

$$\delta Z = \left\{ \int d^3x G \xi, Z \right\}, \quad (10)$$

where ξ is the parameter of the transformation and after computation of the Poisson bracket in the r.h.s. of (10) all the variables should be replaced by their background values.

In our case the constraints (2) – (4) generate the following transformations which will be called for short Lorentz, diffeomorphism and time-evolution respectively.

$$\delta_L B_i^a = i D_i \xi^a \quad (11a)$$

$$\delta_L H_a^i = i \varepsilon_{abc} E_b^i \xi^c \quad (11b)$$

$$\delta_D B_i^a = -i F_{ij}^a \xi^j \quad (12a)$$

$$\delta_D H_a^i = i D_k \left(E_a^{[i} \xi^{k]} \right) \quad (12b)$$

$$\delta_T B_i^a = -2i \xi \varepsilon_{abc} E_b^k F_{ik}^c - i \Lambda \xi \varepsilon^{abc} \varepsilon_{ijk} E_b^j E_c^k \quad (13a)$$

$$\delta_T H_a^i = i D_k \left(\xi \varepsilon_{abc} E_b^{[i} E_c^{k]} \right), \quad (13b)$$

where we used that on the background (6) $F_{ij}^a = -\varepsilon_{ijk} E^{ak}$.

One can see that the action of time-evolution transformation on the the connection fluctuation B_i^a is the addition of an arbitrary proportional to E_a^i contribution. Hence the gauge freedom (13a, b) can be completely fixed by imposing the following condition on B

$$\text{tr } B \equiv B_i^a E_a^i = 0. \quad (14)$$

The diffeomorphism transformation (12a) add an arbitrary antisymmetric term to the connection B . This part of gauge freedom can be eliminated by the following condition

$$(\text{Asym} B)^{ab} \equiv E^{[a} B_i^{b]} = 0. \quad (15)$$

After imposing both conditions (14) and (15) the perturbative scalar constraint (9) is reduced to

$$\text{tr } H \equiv H_a^i E_i^a = 0. \quad (16)$$

Note that in the expression $D_{[i} B_{j]}^a$ the Yang-Mills covariant derivative D_i can be replaced by totally covariant derivative ∇_i with background Christoffel connection. The background triad E and the tensor ε commute with this derivative.

The local Lorentz transformation (11a) shifts the connection fluctuation B_i^a by a gradient term. Thus a natural gauge fixing condition is

$$\nabla^i B_i^a = 0. \quad (17)$$

Note that due to the condition (15) the operator ∇^i in (17) is real. However, unlike the previous cases, this gauge freedom can not be eliminated completely by the last condition. Indeed, undergoing (14) and (15) the B is proportional to a traceless symmetric tensor. The transformation (11a) of traceless symmetric tensor can be written as

$$\delta_L (e^{ic} B_i^a) = \frac{i}{2} \left(\nabla^a \xi^c + \nabla^c \xi^a - \frac{2}{3} \delta^{ac} \nabla_b \xi^b \right), \quad (18)$$

where e^{ic} is the unweighted background triad and $\nabla^a \equiv e^{ai} \nabla_i$. The operator in the r.h.s. of (18) has zero modes of the form

$$\xi_{(0)}^c \sim f^J(t) e^{ci} v_i^J(x_1, x_2, x_3), \quad (19)$$

where v_i^J are the ten conformal Killings vectors of S^3 and $f^J(t)$ are arbitrary functions.

Being imposed the conditions (14) and (15) reduce the vector constraint G_i (8) to

$$G_i = \varepsilon_{ijk} H_a^j E^{ak} = 0. \quad (20)$$

This means that the matrix $H_a^j E^{ak}$ constructed from the triad fluctuations is symmetric. The Gauss law (7) immediately leads to the transversality condition on the triad fluctuations

$$D_i H_a^i = 0. \quad (21)$$

Because of the condition (20) the covariant derivative D_i contains only real part.

The complex reduced phase space consists of symmetric transverse traceless fluctuations of the triad and symmetric transverse traceless fluctuations of the connection. Let us postpone for a while the discussion on the gauge freedom (19) which still remains unfixed, and study the reality conditions (5). After linearization they give

$$H^i{}_a = H_a{}^{i*} \quad B^a{}_i + B_i{}^{a*} = \frac{i}{2} \varepsilon_{abc} \left(e^{k[b} \nabla_{[i} h^c]_k + e^{bj} e^{ck} e_{di} \nabla_{[k} h_j]{}^d \right), \quad (22)$$

where $h^c{}_k$ is the unweighted triad fluctuation. We are to verify that these conditions do not destroy the structure of the reduced phase space. All the conditions (14) – (17), (20) and (21) have the form of a linear real operator acting on fluctuations. Real and imaginary parts are restricted independently. Hence the real part of H and the imaginary part of B satisfy the same symmetry, tracelessness and transversality conditions. As for the second equation (22), one should verify that the r.h.s. of this equation will automatically be symmetric, traceless and transverse. This can be done by straightforward computation.

Consider the gauge freedom (19). It is easy to see that it can not be eliminated by imposing any condition on variables of the space defined by (14) – (17), (20) and (21). This situation is similar to that in ADM gravity [9] and QED [10] on de Sitter space. This gauge freedom can be fixed by imposing the following condition on Lagrange multiplier N^a :

$$\int d^3x e N^a(x, t) e^i{}_a v_i{}^J(x) = 0 \quad (23)$$

for all ten conformal Killing vectors $v_i{}^J$, $J = 1, \dots, 10$. Under the action of Lorentz transformation the variation of N^a looks like following

$$\delta N^a = -i \partial_i \xi^a. \quad (24)$$

Let us find the zero modes of transformation (24) on the space (19). To this end let us calculate the scalar product $\langle \delta N^{a*}, \delta N^a \rangle$

$$\langle \delta N^{a*}, \delta N^a \rangle = \int d\tau \int d^3x e \delta N^a(\xi_{(0)})^* \delta N^a(\xi_{(0)}). \quad (25)$$

This can be done more easily after formal continuation of the integral in the r.h.s. of (25) to Euclidian space with the S^4 metric: $ds^2 = d\tau^2 + \sin^2(\tau) d^2\Omega$, where all relevant operators have discrete spectrum. Performing integration over x and integration by parts over τ and assuming Killing vectors v^J to be orthonormal we obtain

$$- \sum_J \int d\tau \sin(\tau) (f^{*J}(\tau)) \left[\partial_\tau^2 + ctg(\tau) \partial_\tau - \frac{1}{\sin^2(\tau)} + 2 \right] (f^J(\tau)). \quad (26)$$

$f(\tau)$ can be expressed as a power series in eigenfunctions of the self-adjoint operator in (26). These eigenfunctions are just the associated Legendre polynomials $P_n^1(\cos(\tau))$, $n = 1, 2, \dots$. The eigenvalues are $-n(n+1) + 2$. We note that there are ten zero modes f^J corresponding to $n = 1$ and $J = 1, \dots, 10$. They are related to ten Killing vectors of S^4 (or de Sitter space in Minkowski signature). We conclude that the condition (24) give complete fixation of gauge freedom up to the Killing vectors of the space-time. The Killing vectors should be anyhow excluded from the gauge group. In covariant gravity they are excluded from the diffeomorphism group.

In the path integral quantization the gauge freedom (19) can be incorporated in the following way. Note that the Lagrange multipliers N^a proportional to conformal Killing vectors of S^3 do not give any new constraints. Hence such multipliers should be excluded from the integration measure by means of δ -function of the conditions (23). A Jacobian factor should also appear in the integration measure. This procedure was considered in more details for ADM gravity on de Sitter Space [6].

In conclusion, let us summarize the results of this paper and give several remarks.

(i) By solving perturbative constraints and fixing (linearized) gauge freedom on de Sitter background we demonstrated that the complex reduced phase space consists of symmetric transverse traceless fluctuations of the triad and symmetric transverse traceless fluctuations of the connection. Our procedure has the nice property that the constraints eliminate the same components of the triad which are excluded in the connection field by the corresponding gauge fixing. Thus no problem could arise with admissibility of gauge fixing.

(ii) Application of reality conditions do not destroy the structure of the reduced phase space. In particular, the real part of the connection constructed from the constrained triad satisfies all gauge conditions.

(iii) A part of gauge freedom can be fixed only by imposing some condition on Lagrange multiplier.

(iv) It is easy to see that the real reduced phase space of Ashtekar's gravity is equivalent to that of ADM quantum gravity on de Sitter background [6].

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