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Soft Modes Contribution into Path Integral

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A method for nonperturbative path integral calculation is proposed. Quantum mechanics as a simplest example of a quantum field theory is considered. All modes are decomposed into hard (with frequencies $\omega^2 > \omega_0^2$) and soft (with frequencies $\omega^2 < \omega_0^2$) ones, ω_0 is a some parameter. Hard modes contribution is considered by weak coupling expansion. A low energy effective Lagrangian for soft modes is used. In the case of soft modes we apply a strong coupling expansion. To realize this expansion a special basis in functional space of trajectories is considered. A good convergency of proposed procedure in the case of potential $V(x) = \lambda x^4$ is demonstrated. Ground state energy of the unharmonic oscillator is calculated.

Path integral formalism [1] is one of the most useful tools to study a quantum field theory. However there is a serious problem to go out of boundaries of a perturbative theory. There are instanton calculations [2], a lattice calculation method [3] and variational approach which can be used in the case of quantum field theory [4] and sometimes it is possible to find nonperturbative exact results using symmetries of a quantum field model [5].

Here we propose an alternative method for nonperturbative path integral computations. All modes are decomposed into hard (with $\omega^2 > \omega_0^2$) and soft (with $\omega^2 < \omega_0^2$) modes where ω_0 is a some parameter. It is clear that when a frequency is enough large then we can consider a potential term as a perturbation and use a conventional pertubative theory. Thus we can find an effective Lagrangian [6] for soft modes using wellknown perturbative theory. To find a calculation procedure for soft modes we assume that the frequencies of these modes are enough small and in the leading approximation we can neglect a kinetic term and all other terms with derivatives in the effective lagrangian and use a strong coupling expansion. To realize a strong coupling expansion a special basis for trajectories in functional space is suggested and in this basis a regular scheme for the soft modes contribution is formulated in the Section 3.

Here we consider quantum mechanics as a simplest example of a quantum field theory. It is possible, that this method can be applied in a quantum field theory but it requires additional investigations, particularly, to take into consideration a renormalization and a gauge invariance (in the case of a gauge theory). Also a problem of a convergency of this procedure is opened and we just demonstrate a respectively good convergency in the case of unharmonic oscillator with a potential $V(x) = \lambda x^4$.

In the next Section, a path integral and a basis in a functional space of trajectories are considered. In the Section 2, we formulate a procedure of nonperturbative calculation of soft modes contribution in the limit of large coupling constant. The soft modes contribution is calculated in the case of quantum mechanics with a potential λx^4. Then in Section 3 we find the first correction due to the kinetic term. Ground state energy of the system is calculated in Section 4. Here we take into account 2-loop effective potential. In Conclusions we discuss uncertanties of the calculations and possibility to use the procedure in other field theories.

In this paper we consider quantum mechanics in euclidean formalism.

I. PATH INTEGRAL

We consider the following path integral [1]

$$< x_f \mid e^{-\hat{H}t_0} \mid x_i >= \mathcal{N}^{-1} \int \mathcal{D}x(t) e^{-\int_0^{t_0} \mathcal{L}(x(t))dt} \tag{1.1}$$

where $\mathcal{L}(x(t)) = \frac{1}{2}(\frac{dx}{dt})^2 + V(x)$, $x(0) = x_i$, $x(t_0) = x_f$, $\hat{H}$ is a hamiltonian of a system, $\mathcal{N}$ is a normalization factor.

Here we are interesting in a lowest state energy ε_0 and it is convenient to consider the limit $t_0 \to \infty$ and to find a trace over x in (1.1), i.e. $x_i(0) = x_f(t_0)$,

$$\mathcal{Z} = \int dx < x \mid e^{-\hat{H}t_0} \mid x >= \int dx < x \mid n > e^{\varepsilon_n t_0} < n \mid x >|_{t \to \infty} \tag{1.2}$$

$$= \int dx \mid \Psi_0(x) \mid^2 e^{-\varepsilon_0 t_0} = e^{-\varepsilon_0 t_0}$$

where ε_n is an energy of n-th state, and ε_0 is the lowest energy of the system. The factor $\mathcal{N}$ is $\int \mathcal{D}x(t) e^{-\int_0^{t_0} \frac{1}{2}(\frac{dx}{dt})^2 dt}$.

In a perturbative theory the following basis for trajectories is used

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e_n(t) \tag{1.3}$$

where $e_n(t) = \frac{1}{\sqrt{t_0}} e^{i\omega_n t}$, $\omega_n = \frac{2\pi}{t_0} n$, $C_n = C_{-n}^*$.

This basis $\{e_n\}$ has the following normalization

$$< e_n \mid e_m >=< e_n^* e_m >= \int_0^{t_0} e_n^*(t) e_m(t) dt = \delta_{mn} \tag{1.4}$$

Therefore PI (1.1) in basis (1.3) has the following form

$$\mathcal{Z} = \mathcal{N}^{-1} \int \prod_{n=-\infty}^{+\infty} \frac{dC_n}{\sqrt{2\pi}} e^{-<\mathcal{L}(\sum_n C_n e_n)>} \tag{1.5}$$

Here we use the denotation: $< f(t) >= \int_0^{t_0} f(t)dt$.

Hard modes are taken into consideration by conventional perturbative theory and after integration over hard modes we obtain a low energy effective Lagrangian for the soft ones. Soft modes are considered in context of a strong coupling expansion and a soft modes kinetic term is considered as a perturbation as well as all other terms with derivatives in the effective Lagrangian. However a computation of this contribution is rather difficult even if we neglect the kinetic term. It is known the way to use a strong coupling expansion in a lattice theory where we should to choose coupling constants and parameters of a lattice to have a correct continuum limit. Here we propose an alternative approach for strong coupling expansion. We do not change a theory but only change a basis in functional space of trajectories:

$$x(t) = \sum_{|n|<N} B_n E_n(t) + \sum_{|n|>N} C_n e_n(t) \qquad (1.6)$$

$$\omega_0 = \frac{2\pi}{t_0} N;$$

$$< E_{m_1} E_{m_2} ... E_{m_n} >= (\omega_0/\pi)^{(n-2)/2} A_n \delta_{m_1 m_2} \delta_{m_1 m_3} ... \delta_{m_1 m_n}$$

where A_n is a some number which depends on a choice of the basis $\{E_n\}$, $n > 1$. (Notice, that two subspaces $\{e_n\}$ $|n| > N$ and E_n $|n| < N$ are not orthogonal to each other.)

The most important feature of the subspace $\{E_n\}$ is the fact that in this subspace there is a factorization of the path integral if we neglect terms with derivatives in the action. It gives us a possibility to apply a strong coupling expansion. Soft modes belong to the subspace $\{E_n\}$ only. But there are hard modes in this subspace too. In the next Section, a regular procedure for calculation of pure soft modes contribution is formulated. Notice, that this basis E_n breaks translational invariance of the path integral. This invariance is restored when we subtract hard modes contribution out of the subspace $\{E_n\}$.

Here we use one of the possible choices for the basis $\{E_n\}$

$$E_n(t) = \frac{1}{\sqrt{\Delta t}} \Theta(t - t_0/2 - n\Delta t) \Theta(t_0/2 + (n+1)\Delta t - t) \qquad (1.7)$$

where $\Delta t = \pi/\omega_0$. It is obviously that in this basis we have $A_n = 1$.

Below we use the following denotations:
greek letters: μ, ν,..$= 0, \pm 1, .., \pm N$;
small letters: m, n,..$= \pm(N+1), \pm(N+2), ..$;
large letters: M, L,..$= 0, \pm 1, .. \infty$

Let us show that

$$Z = \int \prod_n \frac{dC_n}{\sqrt{2\pi}} \prod_\mu \frac{dB_\mu}{\sqrt{2\pi}} e^{-<\mathcal{L}(\sum(C_n e_n + B_\mu E_\mu))>} \mid J \mid \qquad (1.8)$$

where

$$J = det(< e_\mu E_\nu >) \qquad (1.9)$$

Using that $E_\mu =< E_\mu e_M > e_M$ we have from (8)

$$Z = \int \prod_n \frac{dC_n}{\sqrt{2\pi}} \prod_\mu \frac{dB_\mu}{\sqrt{2\pi}} e^{-<\mathcal{L}(\sum_n C_n e_n + \sum_\mu B_\mu \sum_M < E_\mu e_M^* > e_M)>} \mid J \mid \qquad (1.10)$$

Then shifting C_n we cancel terms with e_n in the sum over M and μ and obtain

$$Z = \int \prod_n \frac{dC_n}{\sqrt{2\pi}} \prod_\mu \frac{dB_\mu}{\sqrt{2\pi}} e^{-<\mathcal{L}(\sum C_n e_n + \sum_\mu \sum_\nu B_\mu < E_\mu e_\nu^* >)>} \mid J \mid \qquad (1.11)$$

Using the following variables

$$C_\mu = \sum_\nu B_\nu < E_\nu e_\mu^* > \qquad (1.12)$$

and taking into account the Jakobian we reproduce eq.(1.5).

Let us calculate $|J|$. The simplest way is to consider the determinant of the following matrix

$$M_{\mu\nu} =< e_\mu^* E_\rho >< E_\rho e_\nu > \qquad (1.13)$$

$$|J| = \sqrt{\det(< e_\mu^* E_\rho >< E_\rho e_\nu >)} \qquad (1.14)$$

Where $M_{\mu\nu}$ is

$$M_{\mu\nu} = \frac{1}{\Delta t t_0} \sum_\rho \int_{\rho\Delta t + t_0/2}^{(\rho+1)\Delta t + t_0/2} e^{-i\omega_\mu t_1} dt_1 \int_{\rho\Delta t + t_0/2}^{(\rho+1)\Delta t + t_0/2} e^{+i\omega_\nu t_2} dt_2 = \qquad (1.15)$$

$$= \frac{(e^{-i\omega_\mu \Delta t} - 1)(e^{+i\omega_\nu \Delta t} - 1)}{\Delta t t_0 \omega_\mu \omega_\nu} \sum_\rho e^{-i(\omega_\mu - \omega_\nu)\Delta t \rho} \qquad (1.16)$$

and the determinant has the following form

$$\det(M_{\mu\nu}) = \prod_{\mu,\nu}\left(\frac{(e^{-i\omega_\mu\Delta t}-1)(e^{+i\omega_\nu\Delta t}-1)}{\Delta t^2\omega_\mu\omega_\nu}\right)\det(N_{\mu\nu}) \quad (1.17)$$

$$N_{\mu\nu} = \frac{\Delta t}{t_0}\sum_\rho e^{-i(\omega_\mu-\omega_\nu)\Delta t} \quad (1.18)$$

There are two different cases for matrix elements $N_{\mu\nu}$: diagonal ($\mu=\nu$) and nondiagonal ($\mu\neq\nu$). When $\mu=\nu$ then we have

$$N_{\mu\nu}\,|_{\mu=\nu} = \frac{\Delta t}{t_0}\sum_\rho 1 = 1 \quad (1.19)$$

Nondiagonal elements are

$$N_{\mu\nu}\,|_{\mu\neq\nu} = \frac{\Delta t}{t_0}\sum_\rho e^{-i(\omega_\mu-\omega_\nu)\Delta t\rho} = 0 \quad (1.20)$$

Here we use the periodical boundary condition for $\{e_\mu\}$

Thus from (1.19) and (1.20) we obtain that

$$\det(N_{\mu\nu}) = 1 \quad (1.21)$$

and

$$|J| = \exp\left(\frac{1}{2}(\sum_{\mu,\nu}\ln(\frac{(e^{-i\omega_\mu\Delta t}-1)(e^{i\omega_\nu\Delta t}-1)}{\Delta t^2\omega_\mu\omega_\nu}))\right) \quad (1.22)$$

$$= \exp\left(\frac{1}{2}\sum_\mu\ln(\frac{2((1-\cos(\omega_\mu\Delta t))}{(\omega_\mu\Delta t)^2}\right)$$

$$= \exp(-\frac{\omega_0 t_0}{2\pi}j)$$

where

$$j = -\int_0^\pi\frac{dx}{\pi}\ln(\frac{2(1-\cos(x))}{x^2}) = 2(\ln(\pi)-1) = 0.289... \quad (1.23)$$

II. SOFT MODES CONTRIBUTION

Let us start to study a quantum mechanics with a potential $V(x)=\lambda x^4$. The Lagrangian has a form

$$\mathcal{L} = \frac{1}{2}(\frac{dx}{dt})^2 + \lambda x^4 \quad (2.1)$$

In terms of our basis $\{E_\mu\}+\{e_n\}$ the path integral is

$$\mathcal{Z} = \frac{1}{N}\int\prod_n\frac{dC_n}{\sqrt{2\pi}}\prod_\mu\frac{dB_\mu}{\sqrt{2\pi}}|J| \quad (2.2)$$

$$\times\exp\{-[\frac{1}{2}|C_n|^2\omega_n^2 + C_n\omega_n^2 < e_nE_\mu > B_\mu + \frac{1}{2}B_\mu < E_\mu e_N > \omega_N^2 < e_N^*E_\nu > B_\nu$$

$$+\lambda(B_\mu^4 < E_\mu^4 > +4B_\mu^3 < E_\mu^3 e_n > C_n + 6B_\mu^2 < E_\mu^2 e_m e_n > C_mC_n$$

$$+4B_\mu < E_\mu e_m e_n e_k > C_mC_nC_k + < e_m e_n e_k e_l > C_mC_nC_kC_l)]\}$$

To cancel linear terms for hard modes (C_n) in the kinetic term we make a shift: $C_n = C_n - < e_n^*E_\mu > B_\mu$. This shift we have made in (1.8). After the shift we have

$$\mathcal{Z} = \frac{1}{N}\int\prod_n\frac{dC_n}{\sqrt{2\pi}}\prod_\mu\frac{dB_\mu}{\sqrt{2\pi}}|J| \quad (2.3)$$

$$\times\exp\{-[\frac{1}{2}|C_n|^2\omega_n^2 + \frac{1}{2}B_\mu < E_\mu e_\rho > \omega_\rho^2 < e_\rho^*E_\nu > B_\nu + \lambda(B_\mu^4 < E_\mu^4 >$$

$$-4B_\mu^3 < E_\mu^3 e_\rho >< e_\rho^*E_\nu > B_\nu + 6B_\mu^2 < E_\mu^2 e_m e_n >< e_m^*E_\nu > B_\nu < e_n^*E_\rho > B_\rho$$

$$-4B_\mu < E_\mu e_m e_n e_k >< e_m^*E_\nu > B_\nu < e_n^*E_\rho > B_\rho < e_k^*E_\lambda > B_\lambda$$

$$+ < e_m e_n e_k e_l >< e_m^*E_\mu > B_\mu < e_n^*E_\nu > B_\nu < e_k^*E_\rho > B_\rho < e_l^*E_\lambda > B_\lambda$$

$$+(\textit{terms with } C_n)]\}$$

In this Section we do not consider the hard modes and neglect kinetic term for soft modes. To calculate the contribution we expand (2.3) over a number of projections from subspace $\{E_\mu\}$ into subspace $\{e_n\}$ and back. This procedure corresponds to a regular subtraction of hard modes out of subspace $\{E_n\}$. These projections decrease a norm of the vector $x(t)$ in a factor $\kappa < 1$ which depends on the vector in functional space. When we integrate over all subspace $\{E_n\}$ we can expect that an effective value of this factor is enough small. To estimate κ we can consider the jakobian $|J|$ which is equal to unit in the case of ortohonality between $\{E_n\}$ and $\{e_\nu\}$ subspaces. A deviation $|J|$ from 1 is a measure of nonortohonality between these two subspaces. In our case $|J| = \exp(-\frac{\omega_0 t_0}{2\pi} j)$ which can be absorbed by rescaling $B_\mu \to B_\mu e^{-j/2}$. It is reasonable to suppose that $\kappa = j/2 = 0.145\ldots$.

Thus, in the leading order of our expansion for soft modes we have

$$\mathcal{Z}_{soft} = e^{-\varepsilon_s^{(0)} t_0} = \frac{1}{\mathcal{N}_{soft}} e^{-\frac{\omega_0 t_0}{2\pi} j_0} \Big(\int_{-\infty}^{+\infty} \frac{dB}{\sqrt{2\pi}} e^{-\lambda B^4/\Delta t} \Big)^{\frac{\omega_0 t_0}{\pi}} \tag{2.4}$$

where

$$\mathcal{N}_{soft} = \int \prod_\mu \frac{dC_\mu}{\sqrt{2\pi}} e^{-\frac{1}{2}|C_\mu|^2 \omega_\mu^2}$$

is the normalization factor for soft modes. In (2.4) we neglect nonortohonality between $\{E_\mu\}$ and $\{e_n\}$ subspaces. In this case $|J| = 1$ and $j = j_0 = 0$.

To take into account the first corrections of the the expansion over numbers of the projections for the jakobian $|J|$ it is useful to represent j in the following form

$$j = -\frac{\pi}{\omega_0 t_0} tr \ln(< E_\mu e_\rho^* >< e_\rho E_\nu >)$$

$$= -\frac{\pi}{\omega_0 t_0} tr \ln(< E_\mu e_N^* >< e_N E_\nu > - < E_\mu e_n >< e_n E_\nu >) \tag{2.5}$$

$$= -\frac{\pi}{\omega_0 t_0} tr \ln(\delta_{\mu\nu} - < E_\mu e_n^* >< e_n E_\nu >)$$

$$= \frac{\pi}{\omega_0 t_0} tr (0 + < E_\mu e_n^* >< e_n E_\nu >$$

$$+ \frac{1}{2} < E_\mu e_n^* >< e_n E_\rho >< E_\rho e_m^* >< e_m E_\nu > + \ldots)$$

$$= j_0 + j_1 + j_2 + \ldots$$

Here

$$j_0 = 0$$

$$j_1 = \frac{2}{\pi} \int_\pi^\infty \frac{1 - \cos(x)}{x^2} dx = 0.227\ldots$$

Thus we have

$$\mathcal{Z}_{soft} = e^{-\varepsilon_s^{(0)} t_0} \tag{2.6}$$

$$\varepsilon_s^{(0)} = \frac{\omega_0}{\pi} (j_0/2 - \ln(\frac{\Gamma(1/4)}{2(4\pi\omega_0\lambda)^{1/4}}) - \ln(\omega_0) + 1) \tag{2.7}$$

$$= \frac{3\omega_0}{4\pi} \left(1 - \ln(\frac{\omega_0 \Gamma(1/4)^{4/3}}{4(\pi e \lambda)^{1/3}})\right)$$

Let us find the first correction for ε_s in the expansion over number of projections from subspace $\{E_\mu\}$ into $\{e_n\}$ and back. It is clear that the first correction appears due to the term $-4\lambda B_\mu^3 < E_\mu^3 e_n >< e_n E_\nu > B_\nu$ in the action for soft modes (2.3). This term gives the following contribution into the path integral

$$\mathcal{Z}_s^{(1a)} = \exp(-(\varepsilon_s^{(0)} + \varepsilon_s^{(1a)}) t_0) = \mathcal{Z}_s^{(0)} (1 - \varepsilon^{(1a)} t_0) = \tag{2.8}$$

$$\frac{1}{\mathcal{N}_s} \int \prod_\mu \frac{dB_\mu}{\sqrt{2\pi}} |J| \exp(-\lambda B_\mu^4 < E_\mu^4 >)(1 + 4\lambda B_\mu^3 < E_\mu^3 e_n >< e_n E_\nu > B_\nu)$$

here we use the denotation $\varepsilon_s^{(1a)}$ because there is another correction of the same order of the expansion.

From (2.8) we obtain

$$\varepsilon_s^{(1a)} = -\frac{1}{t_0} \times \tag{2.9}$$

$$\times \sum_\mu \left(\int \frac{dB_\mu}{\sqrt{2\pi}} (4\lambda B_\mu^3 < E_\mu^3 e_n >< e_n^* E_\nu > B_\nu) \right) \left(\int \prod_\mu \frac{dB_\mu}{\sqrt{2\pi}} \right)^{-1}$$

$$= -4\frac{\omega_0}{\pi}\frac{\Gamma(5/4)}{\Gamma(1/4)}j_1$$

where j_1 is determined in (2.6).

Let us consider a contribution of the following term in the action (2.3):

$$6\lambda B_\mu^2 < E_\mu^2 e_m e_n >< e_n^* E_\nu > B_\nu < e_m^* E_\rho > B_\rho \quad (2.10)$$

This term gives the following correction for ε_s

$$\varepsilon_s^{(1b)} = \frac{\omega_0}{\pi}6\lambda j_1\frac{\omega_0}{\pi}(\int\frac{dB}{\sqrt{2\pi}}B^2 e^{-\lambda B^4\frac{\omega_0}{\pi}})^2(\int\frac{dB}{\sqrt{2\pi}}e^{-\lambda B^4\frac{\omega_0}{\pi}})^{-2} = \quad (2.11)$$

$$\frac{\omega_0}{\pi}6\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\right)^2 j_1 = \frac{\omega_0}{\pi}(0.685...)j_1$$

Thus we have

$$\varepsilon_s \simeq \varepsilon_s^{(0)} + \varepsilon_s^{(1a)} + \varepsilon_s^{(1b)} \quad (2.12)$$

$$= \frac{3\omega_0}{4\pi}\left(1 - \ln(\frac{\omega_0\Gamma(1/4)^{4/3}}{4(\pi e\lambda)^{1/3}e^{0.056}})\right)$$

where the first corrections of our expansion (j_1, $\varepsilon_s^{(1a)}$ and $\varepsilon_s^{(1b)}$) are taking into account in a factor $e^{0.056}$. All other terms of the action (2.3) correspond to the higher corrections of the expansion.

Thus, the next to the leading order of the expansion gives a small contribution into the ε_s ($\sim 6\%$) and we can expect that next corrections of the expansion are small.

The maximal value for ε_s is

$$\varepsilon_s = \frac{3\omega_0}{4\pi} = 0.35\lambda^{1/3};\ at\ \ \omega_0 = \omega^* = e^{0.056}\left(\frac{64\pi e\lambda}{\Gamma^4(1/4)}\right)^{1/3} \simeq 1.55\lambda^{1/3} \quad (2.13)$$

The dependence of $\varepsilon_s(\omega_0)$ on ω_0 is depicted in Fig.1 where we put $\lambda = 1$. The exact value for ground state energy is $0.66..\lambda^{1/3}$. which is about two times larger than the maximal value for ε_s.

III. SOFT MODES KINETIC TERM CONTRIBUTION

Let us take into account a leading contribution of a kinetic term for the soft modes into ε_s. Then this correction for the energy ε_s is

$$\varepsilon_s^{k1} = \frac{\omega_0}{\pi}\left(\frac{1}{2}\int dBB^2 < E_1 e_\rho > \omega_\rho^2 < e_\rho^* E_1 > e^{-\lambda B^4\frac{\omega_0}{\pi}}\right) \quad (3.1)$$

$$\times\left(\int dBe^{-\lambda B^4\frac{\omega_0}{\pi}}\right)^{-1}$$

where

$$< E_1 e_\rho > \omega_\rho^2 < e_\rho^* E_1 >= \sum_\rho \frac{\omega_\rho^2}{t_0\Delta t}\int_0^{\Delta t} e^{i\omega_\rho t_1}dt_1 \int_0^{\Delta t} e^{-i\omega_\rho t_2}dt_2 \quad (3.2)$$

$$= \sum_\rho \frac{\omega_0}{\pi}\frac{\omega_\rho^2}{t_0}\frac{2(1-\cos(\omega_\rho\Delta t))}{\omega_\rho^2}$$

$$= \frac{\omega_0}{\pi}\int_0^{\omega_0}\frac{d\omega}{\pi}2(1-\cos(\frac{\omega}{\omega_0}\pi)) = 2\left(\frac{\omega_0}{\pi}\right)^2$$

Then from (3.1) and (3.2) we have

$$\varepsilon_s^{k1} = \frac{\omega_0}{\pi}\left(\frac{1}{2}\frac{2\omega_0^2}{\pi^2}\sqrt{\frac{\pi}{\omega_0\lambda}}\frac{\Gamma(3/4)}{\Gamma(1/4)}\right) \quad (3.3)$$

$$= \frac{\omega_0}{\pi}\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\sqrt{\frac{\omega_0^3}{\pi^3\lambda}}\right)$$

At $\omega_0 = \omega^*$ we have

$$\varepsilon_s^{k1} \simeq 0.14\varepsilon_s \quad (3.4)$$

The next correction at $\omega_0 = \omega^*$ is about 1% and we do not take it into consideration.

IV. THE GROUND STATE ENERGY

To have a reliable result for the ground state energy we need to take into consideration the hard modes contribution. We should integrate over hard modes using loop expansion and find a low energy effective Lagrangian. Here we consider the leading order of the expansion over number of projections from the subspace $\{E_\mu\}$ into the subspace $\{e_n\}$ and back. It was shown in Section 3 that an uncertainty of this approximation is about a few percents at $\omega_0 = \omega^*$ and we expect that an accuracy of our calculations will be about few percents in this leading approximation. From (2.2) and (2.3) we see that in this approximation there is no linear terms for hard modes in the action. It is easy to find one-loop effective potential for soft modes:

$$V^{(1)}(x_s) = \frac{1}{2\pi}\left(\sqrt{12\lambda x_s^2}(\pi - 2\arctan(\frac{\omega_0}{\sqrt{12\lambda x_s^2}})) - \omega_0 \ln(1 + \frac{12\lambda x_s^2}{\omega_0^2})\right) \quad (4.1)$$

Let us calculate the soft modes contribution into the energy using 1-loop effective potential (4.1)

$$\varepsilon^{1-loop}(\omega_0) = \frac{\omega_0}{\pi}\left(1 - \ln(\omega_0\sqrt{\frac{2}{\pi}}\int_0^\infty dB e^{-\frac{\pi}{\omega_0}V(B\sqrt{\frac{\omega_0}{\pi}})})\right) \quad (4.2)$$

here $V(x) = V^{1-loop}(x) = \lambda x^4 + V^{(1)}(x)$.

The dependence of ε^{1-loop} on ω_0 is depicted in Fig.1 (line (b)). From Fig.1 we see that 1-loop hard mode contribution is comparable with ε_s. Line (c) in Fig.1 shows the dependence of $\varepsilon^{1-loop}(\omega_0) + \varepsilon_s^{k1}(\omega_0)$. Where ε_s^{k1} is the leading kinetic term contribution. And line (d) in Fig.1 corresponds to $\varepsilon^{2-loop}(\omega_0)$ which is calculated according eq.(4.2) with 2-loop effective potential $V(x)$ for soft modes where $V(x) = V^{2-loop}(x) = V^{1-loop}(x) + V^{(2)}(x)$ and the leading kinetic term contribution is taken into account. The potential $V^{(2)}(x)$ has the following form:

$$V^{(2)}(x) = \frac{1}{4\pi^2 x^2}\left(\frac{\pi}{2} - \arctan(\frac{\omega_0}{\sqrt{12\lambda x^2}})\right)^2 \quad (4.3)$$

$$-48\lambda^2 x^2 \int \frac{d\omega_1 d\omega_2 d\omega_3 \delta(\omega_1 + \omega_2 + \omega_3)}{(2\pi)^2(\omega_1^2 + 12\lambda x^2)(\omega_2^2 + 12\lambda x^2)(\omega_3^2 + 12\lambda x^2)}$$

where $\omega_1^2 > \omega_0^2$, $\omega_2^2 > \omega_0^2$, $\omega_3^2 > \omega_0^2$.

In Fig.1 (line (d)) we see a very weak dependence of $\varepsilon^{2-loop}(\omega_0)$ on parameter ω_0 in a large region ($\lambda^{1/3} < \omega^0 < 2.5\lambda^{1/3}$). In this region the value of the next

to the leading corrections is an order of variation $varepsilon(\omega_0)$. and $\varepsilon^{2-loop} \simeq (6.8 \pm 0.3)\lambda^{1/3}$ which is in a good agreement with exact result: $\varepsilon = 0.66...$ (curve (e) in Fig.1). Thus we see a selfconsistence of the expansion in question.

V. DISCUSSION

Let us discuss the main features of the approach. Two main assumptions are used here. The first one is that we suppose that an expansion over the numbers of projections from $\{E_\nu\}$ to $\{e_n\}$ and back does not diverge. It was shown that the first correction of the expansion is rather small in the case of the potential λx^4 but the general structure of this expansion is not known. The second assumption is that there is a region for the parameter ω_0 where a perturbative expansion for hard modes and a strong coupling expansion for soft mode work at a same time. The results obtained have shown a correctness of these assumptions in the case of the potential λx^4. However, it is clear that this method does not work for a potential which does not tend to infinity at $x \to \pm\infty$. Also it is not possible to use this method (at least directly) in instanton case due to the large kinetic term corrections in soft modes sector. However this method gives a reasonable results in the case of the potential considered here.

The next important problem is a question on translation invariance. It is clear that this invariance is broken when we use basis E_ν. However in Section 1 it was shown that the path integral in this basis is equal to the path integral in the basis e_n which does not break translational invariance. The expansion over a numbers of projections from subspace $\{E_\nu\}$ into $\{e_n\}$ and back corresponds to subtraction of translational noninvariant contributions. In the case when this expansion works we can control these contributions.

Here we considered the ground state energy only. This parameter is not convenient to study a restoration of translational invariance. In this context it is interesting to investigate a propagator $< x(t_1)x(t_2) >$. This question is very important for understanding of applicability of the procedure. The propagator has the following form

$$S(t_1, t_2) = \ll x(t_1), x(t_2) \gg \quad (5.1)$$

$$= \ll B_\mu B_\nu \gg (E_\mu(t_1) - < E_\mu e_m^* > e_m(t_1))(E_\nu(t_2) - < E_\nu e_n^* > e_n(t_2))$$

$$+ \ll C_m C_n \gg e_m(t_1) e_n(t_2)$$

Here we take into account the shift $C_n \to C_n - B_\nu < E_\nu e_n >$ which was introduced in the first Section, $\ll\gg$ denotes the average value for the path integral (see (2.2,2.3)). In eq.(5.1) we use that in the leading order of the expansion over numbers of projections $\ll B_\mu C_n \gg = 0$. It is obvious that $\ll C_n C_m \gg \sim \delta_{nm}$ and the second term in eq.(5.1) depends on $(t_1 - t_2)$ only and does not break translational invariance. In the leading order we have that $\ll B_\mu B_\nu \gg \sim \delta_{\mu\nu}$. Then using (1.19) and (1.20) we obtain that the first term in (5.1) depends on $(t_1 - t_2)$ only also. It can be shown that next to the leading corrections of the expansion do not break the translational invariance. Probably, that this procedure does not break the invariance in any order of the expansion.

The most interesting application of the method is quantum field theory. In this case a renormalization should be taken into consideration by a standard way in an effective Lagrangian. For a gauge theory it is necessary to study a question on a gauge invariance.

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Figures

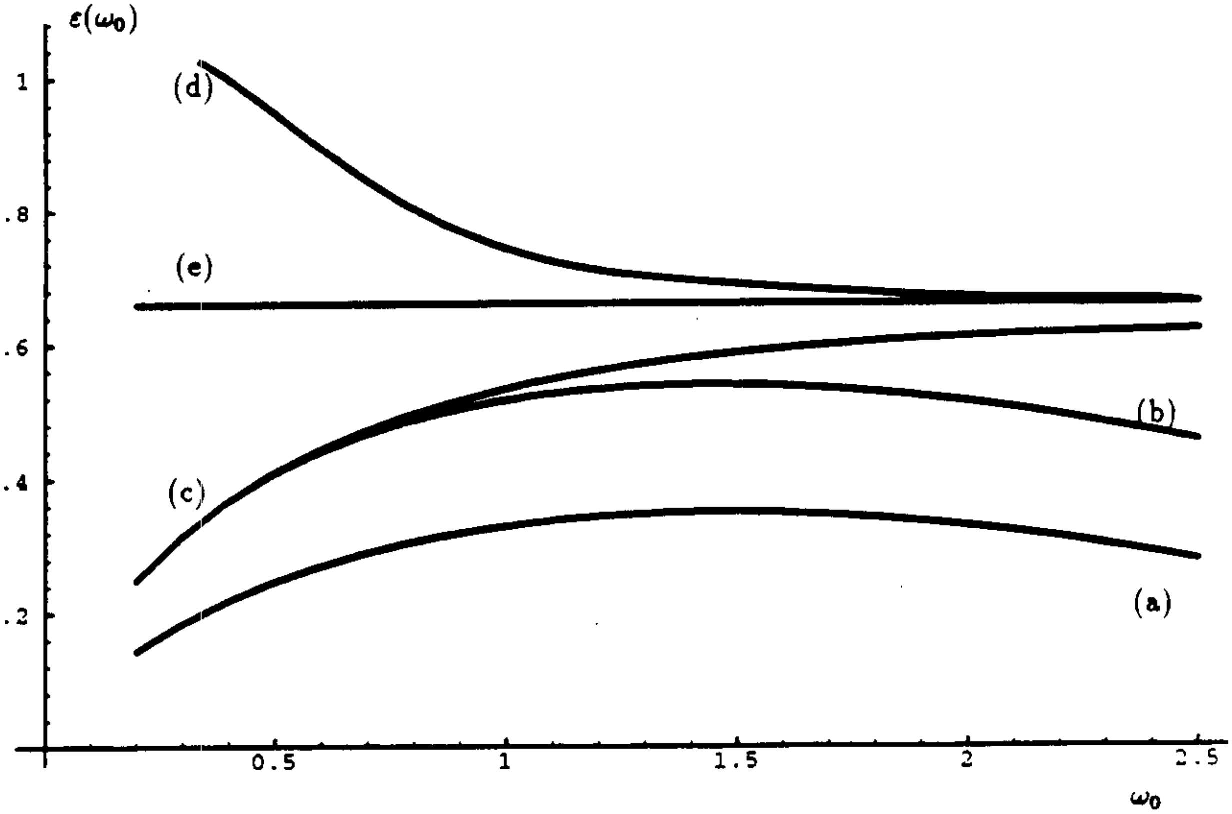

Fig.1