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Comment regarding bounds upon heavy meson form factors

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We present a counterexample to a recently proposed bound on the Isgur-Wise function. We also discuss which of the assumptions underlying the bound is faulty, and present a physical interpretation of an experimental measurement which would violate the bounds.

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I. INTRODUCTION

Recently, a lower bound was proposed [1] for the universal Isgur-Wise function [2] ξ that describes "elastic" heavy meson form factors. The bound is striking because, when combined with Bjorken's upper bound [3], the behaviour of ξ would be very highly constrained. For example, with the kinematics relevant to $\bar{B} \to De\bar{\nu}_e$ decay, these bounds prohibit ξ from falling by less than 12% or more than 21% from minimum to maximum recoil. The lower bound conflicts with many form factor models [4–7], and could have considerable impact upon such matters as the extraction of Cabibbo-Kobayashi-Maskawa parameters from data.

In this paper we present a counterexample to the lower bound [1]. Using the counterexample, we identify which of the hypotheses needed to prove this bound fails, and in doing so discover an interesting physical interpretation of an experimental violation of this bound.

For orientation, we first review the proof.

II. THE BOUNDS

The upper bound for ξ is given in [3], but current interest centers on a lower bound derived by de Rafael and Taron [1]. Three hypotheses underlie the proof. The first is an integral constraint on the magnitude of the form factor in the production region; we begin by briefly racapitulating the derivation of that constraint.

By heavy quark symmetry, $\xi(w) = F(2m_b^2[1-w])$ where $w = v \cdot v' = p \cdot p'/m_b^2$ and $F(q^2)$ is the elastic form factor defined using $V^\mu = \bar{b}\gamma^\mu b$ and

$$< \bar{B}(p')|V^\mu|\bar{B}(p) >= F(q^2)(p+p')^\mu, \tag{1}$$

where $q = p - p'$. The derivation of the constraint starts by using a dispersion relation for the scalar function defined from the correlator

$$\begin{aligned} \Pi^{\mu\nu} &= \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) \Pi(q^2) \\ &= i \int d^4x e^{iqx} < 0|T[V^\mu(x)V^\nu(0)]|0 > . \end{aligned} \tag{2}$$

The constraint on $|F|^2$ is obtained by comparing two evaluations of $\chi = \partial\Pi/\partial q^2$. Ref. [1] first evaluates χ perturbatively as a $\bar{b}b$ loop, using its value at $q^2 = 0$ where the

perturbative and non-perturbative corrections are expected to be small [1]. They also write a dispersion relation for χ, and evaluate $Im\Pi$ by inserting a sum over a complete set of hadronic intermediate states in (2). All contributions to $Im\Pi$ are positive, so that keeping only the $B\bar{B}$ contribution must underestimate the sum, leading to a bound

$$J_{B\bar{B}} \equiv \frac{5}{48}\int_1^\infty dy y^{-7/2}(y-1)^{3/2}\,|F(y)|^2 \leq 1, \tag{3}$$

where $y \equiv (1-w)/2 = q^2/4m_B^2$. A stronger condition can be obtained by including the $B^*\bar{B}$, $B\bar{B}^*$, and $B^*\bar{B}^*$ states which depend upon the same form factor by heavy quark symmetry; the corresponding integral J must also be less than or equal to unity.

The three conditions for obtaining the bound on F are:

1) $J \leq 1$,

2) $F(0) = 1$,

3) $F(y)$ is analytic except for the cut along $1 \leq y < \infty$.

Given these conditions, the proof begins by mapping the integral giving J into an integral around the unit circle using the transformation [1, 8],

$$i\frac{1+z}{1-z} = \sqrt{y-1} \quad . \tag{4}$$

If $(y-1) \equiv re^{i\phi}$, then the sheet $0 < \phi < 2\pi$ is mapped to the interior of the unit circle. The region $-\infty < y \leq 0$ maps to the real line $0 \leq z \leq 1$. If $F(z)$ is analytic, it is clear that one can use Cauchy's theorem to get the value of F inside the unit circle from its value on the boundary. What is interesting is that an upper bound for $|F|^2$ on the unit circle gives both lower and upper bounds on F anywhere inside the circle.

To see this, define an inner product

$$< g|f >= \frac{1}{2\pi}\int_0^{2\pi} d\theta g^* f. \tag{5}$$

In terms of this inner product, one can rewrite the bound on J as

$$J = \frac{1}{2\pi}\int_0^{2\pi} d\theta\,|\phi(z)F(z)|^2 = ||\phi F||^2 \leq 1. \tag{6}$$

For the bound on J_{BB} corresponding to (3), $\phi(z) = (1/16)\sqrt{5\pi/6}(1+z)^2\sqrt{1-z}$. Two useful vectors in this Hilbert space of functions are [8]

$$g_1 = 1 \tag{7}$$

and

$$g_2 = \frac{1}{1-\lambda^* z}, \tag{8}$$

so that, using Cauchy's theorem,

$$< g_1|\phi F >= \phi(0) \tag{9}$$

and

$$< g_2|\phi F >= \phi(\lambda)F(\lambda) \quad . \tag{10}$$

It is here that the assumed analyticity of F, which translates into $F(z)$ being analytic inside the unit circle, is crucial.

To complete the proof, note that

$$||\phi F + \mu g_1 + \rho g_2||^2 \geq 0 \tag{11}$$

for any values of μ and ρ. The left hand side of the above expression, minimised with respect to μ and ρ, is a quadratic function of $F(\lambda)$ with everything else defined or calculable. Positivity then requires that $F(\lambda)$ lie between certain definite upper and lower bounds, which are the bounds given in Ref. [1]. The upper bound is weaker than Bjorken's bound [3], but the lower bound is very restrictive. In terms of the "charge radius" defined by

$$\frac{d\xi}{dw}(0) \equiv -\rho^2, \tag{12}$$

the combination of Bjorken's bound and the lower bound of Ref. [1] which arises from J gives

$$\frac{1}{4} \leq \rho^2 \leq 0.48. \tag{13}$$

III. A COUNTEREXAMPLE

We can see the flaw in the preceeding argument most simply in a system to which the bound should apply, but for which it is manifestly false. This system is a $b\bar{c}$ meson (which for the purposes of this argument may be idealized to a system in which m_b, $m_c \to \infty$ but $m_c \ll m_b$, although the arguments we present should be approximately valid for the real $\bar{B}_c$ meson). The ground state of this system will to a high degree of accuracy behave like a nonrelativistic Coulombic bound state with "Bohr radius" $\frac{4}{3}m_c\alpha_s$, and it would therefore have

$$\rho^2 = \frac{9}{16\alpha_s^2} \tag{14}$$

which clearly doesn't satisfy the upper bound in (13), corresponding to the lower bound on the Isgur-Wise function derived in Ref. [1]. The flaw in the proof of the bound contains some interesting and possibly useful physics.

Consider first the approximation of the idealized $\bar{B}_c$ meson as hydrogenic. This approximation will lead to an arbitrarily accurate description of ξ as m_b, $m_c \to \infty$ with $m_c \ll m_b$. In this approximation the $\bar{B}_c$ is nonrelativistic with a wavefunction proportional to $\exp(-r/a_0)$, where $a_0^{-1} = \frac{4}{3}m_c\alpha_s$, and the form factor for scattering off the b quark is given under these conditions by

$$F = \int d^3r e^{-i\frac{m_b}{m_c}\vec{q}\cdot\vec{r}}|\Psi(r)|^2 = \left(1 + \frac{9|\vec{q}|^2}{64m_b^2\alpha_s^2}\right)^{-2}. \tag{15}$$

Thus the form factor has a pole in the timelike region at $q^2 = -|\vec{q}|^2 = \frac{64}{9}m_b^2\alpha_s^2$, well below the threshold for $B\bar{B}$ production. In this approximation to the $\bar{B}_c$ system, the assumed analytic structure of $F(q^2)$ is not realized.

While this approximation is suggestive of the flaw in the proof given in [1], and while we believe it to be physically accurate (*i.e.*, it shows that the failure of the bound has its origin in its neglect of the composite structure of the mesons), it leaves something to be desired. (These basic arguments were given much earlier in this context in Refs. [4, 9, 10], in which the existence of such "unexpected" charge radii was traced to the presence of "pseudo-anomalous-thresholds", that is, large but not quite singular structures which occur near to where the anomalous cut would be in the absence of confinement.) The problem is that the singularity in $F(q^2)$ as given

in (15) arises from the large r behavior of the wavefunction, which should, of course, not be Coulombic because of confinement. The "real" $F(q^2)$ will therefore be sharply peaked near $q^2 = \frac{64}{9}m_b^2\alpha_s^2$ (it will display a "pseudo-anomalous-threshold"), but not actually singular. Thus, while the counterexample stands, it leaves the flaw in the proof of the bound unclear.

Although equally unsatisfactory in its own way, we note in passing that a harmonically confined nonrelativistic system has a gaussian form factor which is analytic everywhere apart from an essential singularity at *infinite* q^2, which is the point $z = 1$ in the transformed space. A sharply peaked $F(q^2)$ would not contravene the theorem, but an essential singularity on the boundary of the unit circle does.

As Jaffe pointed out [11] in the context of a potential model, neither of these pictures gives a satisfactory picture of the analytic structure of $F(q^2)$. The complete wavefunction of a confined system will contain not only the highly damped wavefunction of the confined constituents, but also the exponentially damped tails of the virtual decay channels of this state. In Jaffe's example, these states produced wildly oscillating singularities along the normal cut ($q^2 \geq 4m_B^2$ in the case at hand) which built up the sharp spike at $q^2 = \frac{64}{9}m_b^2\alpha_s^2$. A field theoretic realisation of this scenario was considered by Jaffe and Mende [12], who showed that in the $N_c \to \infty$ limit of 1+1 dimensional QCD , the pseudo-anomalous-threshold in a world with a single flavor of quark is built up by a series of narrow resonances with wildly oscillating residues. These more realistic descriptions present us with a puzzle: the lower bound fails, but there do not seem to be any singularities near $q^2 = \frac{64}{9}m_b^2\alpha_s^2$ to violate the theorem.

A partial resolution of this puzzle emerges once one notices that the form factor of the $\bar{B}_c$ meson will in fact have *real* anomalous thresholds. Recall that such anomalous singularities arise from triangle diagrams like that shown in Fig. 1. The results are well known [13]. For very strong binding, meaning $m_{B_c}^2 < M_b^2 + M_c^2$, such diagrams have cuts only for real q^2 satisfying $q^2 \geq 4m_B^2$. This is the "normal threshold". However, if $m_{B_c}^2 > M_b^2 + M_c^2$, the cut begins at the "anomalous threshold,"

$$q^2_{anom} = -\frac{\left(m_{B_c}^2 - (M_b + M_c)^2\right)\left(m_{B_c}^2 - (M_b - M_c)^2\right)}{M_c^2} \approx \frac{8M_b\,(M_b + M_c)\,\epsilon}{M_c}, \tag{16}$$

where in the second form we define the binding energy by $m_{B_c} = M_b + M_c - \epsilon$, and keep only the leading term in ϵ. For the Coulombic case, this threshold is the same as the pole location in the form factor of eq. (15). This singularity has an origin that is easy to understand in nonrelativistic perturbation theory. Consider, *e.g.*, the positronium ground state system at rest. The nonrelativistic version of the singularity of Fig. 1 corresponds to the fact that this state can dissociate virtually into an e^+e^- pair with imaginary momenta $\vec{p} = \pm\frac{i}{2a_0}\hat{e}$ (where $\hat{e}$ is a unit vector) so that $E_+ + E_- = 2m_e - 1/4m_e a_0^2$ is equal to the ground state energy, *i.e.*, the intermediate state is on the energy shell so that the virtual process is singular. A probe for the electron bringing in momentum $\vec{q} = +\frac{2i}{a_0}\hat{e}$ and energy $\omega = -|\vec{q}|^2/4m_e$ will then produce an elastic scattering via another on-energy-shell intermediate state. The associated form factor will have a singularity at $|\vec{q}|^2 = 4/a_0^2$ as expected.

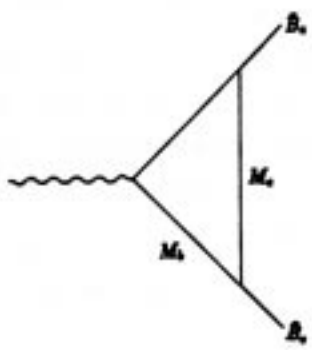

Fig. 1: A typical triangle graph which can have anomalous threshold singularities; M_b and M_c are the masses of states coupled to $\bar{B}_c$, which can represent $\bar{b}$- and $\bar{c}$-containing hadrons or b and $\bar{c}$ quarks.

In the examples considered in Refs. [11,12], there is only a single flavor of quark, so that the states considered are of the form $Q\bar{Q}$ and their lowest virtual decay channel is $Q\bar{Q} \to Q\bar{Q} + Q\bar{Q}$. Thus $m^2 < m_1^2 + m_2^2$ and there is no anomalous threshold. However, in a world containing light quarks, our $\bar{B}_c$ would have open virtual decay channels starting with $\bar{B}_c \to \bar{B} + D$, satisfying $m_{B_c}^2 > m_B^2 + m_D^2$ so that there *are* anomalous thresholds. Indeed, one can easily show that in the limiting case the resulting singularities are exactly those required to produce the pole seen in Eq. (15).

This behavior has an interpretation in terms of duality: all of the dissociation channels have a mass which in the limiting case are at the "hydrogenic" threshold with respect to m_{B_c} and thus contribute *hadronic* exponential tails which produce the singularities.

Although this discussion means that we have identified an illegal assumption of the proof of Ref. [1], it cannot be the full story since the violation of the bound by $\bar{B}_c$ should be independent of the existence of light quarks. However, if c is the lightest quark, then the lowest virtual decay channel of $\bar{B}_c$ would be $\bar{B}_c + \eta_c$ and there would be no anomalous thresholds! Since it is this case that corresponds most closely to the case of the $\bar{B}$ meson, it too must be addressed.

We should not find the elusive nature of the flaw in the proof surprising. The examples we have seen tell us that the physics is in the "charge radius" of the hadron and not in the positions of the singularities which can move from $q^2 = \frac{64}{9} m_b^2 \alpha_s^2$ for the hydrogenic case (which we have seen includes $\bar{B}_c$ in the presence of light quarks), to $q^2 = \infty$ for the case of naive confinement, to a sequence of very strong poles of oscillating residues near $q^2 = 4m_B^2$. Thus, although the assumed analytical structure of $F(q^2)$ is a technical flaw of our counterexample, we have not yet completed the delineation of the the flaw of the proof, since $\bar{B}_c$ stands as a counterexample even in a world in which charm is the lightest quark (so that there are no anomalous thresholds). As pointed out to us by Falk, Luke, and Wise [14] (see also [15, 16]), in this case as well the flaw can be found in the assumed analytic structure of F. In particular, the Υ or $b\bar{b}$ bound states that lie below the threshold for $B\bar{B}$ production will give poles in the form factor F which lie within the unit circle in the z-plane as previously defined, and will violate one of the conditions required for the proof of the bounds. That these poles are implicated in the failure of the bound is certainly consistent with the observations of Ref. [12]. The dependence of the residues of these poles on the c (or in general, lighter) quark mass is also consistent with the requirement that they reflect the structure of the $\bar{B}_c$ bound state even though they themselves are $b\bar{b}$ bound states.

The z-plane is shown in Fig. 2 with some corresponding points in $y = q^2/4m_B^2$ indicated. Each value of y (except $y = 1$ and ∞) maps into two values of z. The physical region for $B\bar{B}$ production, the line $1 \leq y < \infty$, maps once into the unit

semicircle above the real axis and once into the unit semicircle below. An Υ that lies above the $B\bar{B}$ threshold would, if its width were zero, give two poles which both lie on the unit circle in the z-plane, at points illustrated by the open circles. Accounting for the width of the Υ moves both of the poles outward and outside the unit circle, and they cause no problems with the proof.

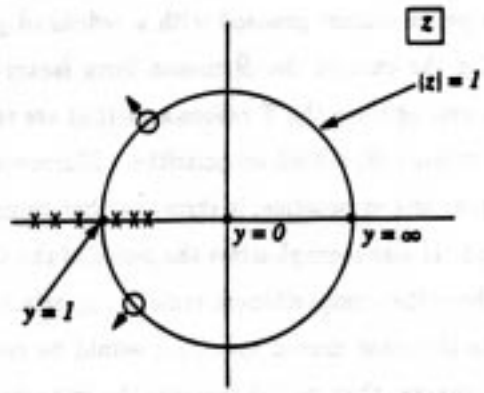

Fig. 2: The mapping $y \to z$. The unit circle $|z| = 1$ contains the real line $1 \leq y < \infty$ twice. A pole on this line gives poles at two locations on $|z| = 1$, indicated by the small open circles. Giving the pole location in the y-plane an imaginary part with the correct sign for a physical resonance would move both poles in the z-plane outside the unit circle. Poles below the $B\bar{B}$ threshold, on the line $0 < y < 1$, give two poles each on the negative real z-axis, one within and one without the unit circle at locations illustrated by the markings "×".

The unphysical region for $B\bar{B}$ production, $0 < y < 1$, maps onto negative real z, once onto z between 0 and -1 and once onto z between -1 and $-\infty$. The three (in the real case) Υ's that lie below $B\bar{B}$ threshold give poles in $F(z)$ at locations illustrated by the marks "×". Three of these lie within the unit circle; accounting for their small width will move them smoothly away from the positions indicated but not outside the unit circle. These singularities thus violate the assumption on the analyticity of the form factor needed to derive (10).

One might wonder if the bounds could be rescued by redefining y into $y' = q^2/m_\Upsilon^2$ and using y' to do the mapping to z. All Υ poles would then lie outside the unit circle (and one would argue that cuts due, for example, to $3\pi \to B\bar{B}$ are too weak to be significant). Consider, however, the constraint $J \leq 1$. J is an integral over $|F|^2$ with a weighting function. The weighting function became the factor $|\phi|^2$ on

the unit circle, with ϕ analytic within the unit circle. The analyticity of $\phi(z)$ is as important as that of F in the proof of the bound. The weighting function came from integrals over a set of real states in the evaluation of $Im\Pi$, and cannot be changed. Implementing it with the variable y' would require a step function $\Theta(y' - 4m_B^2/m_\Upsilon^2)$, which is not analytic and could not later be replaced by a $|\phi|^2$ on the unit circle with ϕ analytic. Hence the proof cannot proceed with a redefined y.

Note that while for the case of the $\bar{B}$ meson form factor it is only the pseudo-anomalous-thresholds arising from the Υ resonances that are relevant, $\bar{B}_s$ mesons will already have real anomalous threshold singularities. Moreover, while the $\bar{B}_s$ system may not be an important one in practice, it shows us that compositeness is the key to the failure of the bound. It also reemphasizes the perils of the use of dispersion theory [11, 12] for systems where the compositeness scale is important: although (at least in the large N_c limit) the $b\bar{b}$ vector meson spectrum would be essentially unchanged by the existence of light quarks, they would transfer the important singularities for describing the $\bar{B}_s$ form factor from these $b\bar{b}$ poles to the $\bar{B}$ D-type anomalous thresholds. It might be interesting to delineate the mechanism by which this tranfer occurs.

IV. SUMMARY

The proof of the lower bound on the Isgur-Wise function requires, among other things, that it be analytic over most of the complex q^2 plane, particularly including the region along the real axis below the normal threshold at $q^2 = 4m_B^2$. We have shown that the form factor of the $\bar{B}_s$ meson would violate this assumption in the Coulombic approximation (*via* an anomalous threshold at the quark level), in a world with naive confinement (*via* an essential singularity), in a world with light quarks (*via* an anomalous threshold at the hadronic level), and in a world in which the charm quark is the lightest quark (*via* the subthreshold vector meson poles in the t-channel which build up the "pseudo-anomalous-threshold"). On the positive side, the bounds of de Rafael and Taron can provide useful information. If these bounds are indeed violated by the data, then this would be evidence for the existence of real or pseudo-anomalous-thresholds. In either case, this would be the first time that such effects would be seen to be important in hadronic physics.

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