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# RELATIVISTIC EFFECTS and RELATIVISTIC METHODS*

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## I. Introduction and Overview

In the past, the vast majority of nuclear physics calculations were carried out using nonrelativistic quantum mechanics. Relativistic effects, to the extent they were considered at all, were usually regarded as small corrections, primarily kinematic in origin. However, as understanding of hadronic matter has developed, and as high energy accelerators capable of probing hadronic systems to very high momenta have become available, interest in relativistic methods has grown and theoretical techniques have matured. Until the early 1980's, most research was centered on methods for computing relativistic corrections to calculations which are essentially non-relativistic. The idea was to find corrections to lowest order in $(v/c)^2$, where v is a typical particle velocity regarded as small compared to nuclear energies and masses. Recent work goes far beyond such expansion methods. Fully covariant approaches, in which the dynamics is closely connected to field theory, are now being developed. Such methods have several advantages over the earily expansion methods. Their close connection to field theory makes it possible to study fundamental issues by applying dynamical models to a wide variety of physical processes in a *consistent* way, and covariance insures that high energy calculations include $(v/c)^2$ effects to all orders.

In this chapter I will review both the early expansion methods and the newer, fully covariant techniques. The former are still in wide use, and are the only methods available for the treatment of complex nuclei. They also provide the student with a reasonable introduction to many of the ideas and issues. The newer covariant methods have been

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applied mainly to few body systems and infinite nuclear matter, but I will only discuss the few body applications. Relativistic nuclear matter and the relativistic many body problem have been treated in a number of books [see, for example, CS86 and SW86].

This chapter is organized into three main Parts, each with several sections. The remainder of this Part is devoted to a review of several processes in which relativistic effects are known to play a role. These include (i) deuteron form factors and static moments, (ii) deuteron photodisintegration, (iii) deuteron electrodisintegration and radiative neutron capture, (iv) three nucleon form factors, and (v) the *NN* interaction. In each case, theoretical results will be stated and compared with experiment. Part I will then conclude with a discussion of the implications of the results presented, and an assessment of the role of relativistic effects in nuclear physics. In Part II, the $(v/c)^2$ expansion methods for calculating corrections to the deuteron electric form factors will discussed in considerable detail. Two different methods, illustrating the different "schools" which have developed, will be presented and compared. It is hoped that this somewhat detailed discussion will provide sufficient background so that a reader unfamilar with this subject can persue the older literature on her own. Sections 2.4 and 2.7 include summaries and discussion of the work in Part II. The last Part will discuss the newer, covariant methods which have been extensively applied to the study of few body systems in the last decade. Since the newer methods are quite different, this last Part is developed in a way which is largely independent of the discussion in the first two Parts. A final conclusion is given in sec. 3.5.

Many of these topics have been reviewed periodically in talks given to the International Conferences on Few Body Problems [Gr76, 84a, Fr81, Tj87, and Po90] and Particles and Nuclei (PANIC) [Gr77], and to the European Conferences on Few Body Problems [Gr84b, 86, Ga86, Tj86c] and Nuclear Physics with Electromagnetic Probes [Fr79b, Pl85, and Tj85]. There is also a discussion by Friar in the book *Mesons in Nuclei* [Fr79a].

## 1.1 Deuteron Form Factors and Static Moments

The deuteron is an important special case because all of the corrections which will be discussed in this chapter have been applied to it, and there are also excellent high $Q^2$

data to compare with theory. Since the deuteron is a spin one particle, it has three form factors, which will be denoted $G_C$, $G_Q$, and $G_M$ for the charge, quadrupole, and magnetic form factors, respectively [AC80]. The differential cross section for unpolarized elastic scattering is

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}\bigg|_{NS} \left[A\left(Q^2\right) + B\left(Q^2\right)\tan^2\left(\tfrac{\theta}{2}\right)\right] \tag{1.1}$$

where $\theta$ is the electron scattering angle, $NS$ is the differential cross section for no structure, and the four momentum transferred by the electron is denoted by $q$, with $q^2 < 0$. [The conventions of BD65 will be used in this chapter, together with the SLAC convention, $Q^2 = -q^2$.] The $A$ and $B$ structure functions are related to the form factors by

$$\begin{aligned} A(Q^2) &= G_C^2(Q^2) + \frac{Q^2}{6M_d^2}\, G_M^2(Q^2) + \frac{Q^4}{18M_d^4}\, G_Q^2(Q^2) \\ B(Q^2) &= \frac{Q^2}{3M_d^2}\left(1 + \frac{Q^2}{4M_d^2}\right) G_M^2(Q^2) \end{aligned} \tag{1.2}$$

where $G_M(0) = \mu_d$ in units of $e/2M_d$ (twice the usual normalization) and $G_Q(0) = Q_d$ in units of $e/M_d^2$.

All three form factors have been the focus of an extensive program of theoretical calculations. In both the relativistic and nonrelativistic cases, the form factors will be written ,

$$\begin{aligned} G_C(Q^2) &= G_E^S(Q^2)\, D_C(Q^2) + \left[2G_M^S(Q^2) - G_E^S(Q^2)\right] D_C^{SO}(Q^2) \\ G_Q(Q^2) &= G_E^S(Q^2)\, D_Q(Q^2) + \left[2G_M^S(Q^2) - G_E^S(Q^2)\right] D_Q^{SO}(Q^2) \\ G_M(Q^2) &= G_E^S(Q^2)\, D_M^E(Q^2) + G_M^S(Q^2)\, D_M^M(Q^2) \end{aligned} \tag{1.3}$$

where $G_E{}^S$ and $G_M{}^S$ are the isoscalar nucleon form factors [normalized to $G_E{}^S(0) = 1$, $G_M{}^S(0) = \mu_S$ ], and the $D$'s are body form factors which depend on the deuteron wave functions. In the non-relativistic impulse approximation (NRIA) the body form factors are

(Ja56, Go63)

$$D_C = F_0 = \int_0^\infty dr\ j_0(\tau)\ C(r) \qquad \left(\frac{Q^2}{6\sqrt{2}\,M_d^2}\right) D_Q = F_2 = \int_0^\infty dr\ j_2(\tau)\ Q(r)$$

$$D_M^M = F_M = 2F_0 + \sqrt{2}\ F_2 + \int_0^\infty dr\left[\,j_2(\tau) - 2\,j_0(\tau)\right]\tfrac{3}{2}\,w^2$$

$$D_M^E = \int_0^\infty dr\left[\,j_0(\tau) + j_2(\tau)\right]\tfrac{3}{2}\,w^2 \tag{1.4}$$

where $C(r)$ and $Q(r)$ are convenient combinations of the reduced $S$ and $D$ state wave functions of the deuteron, $u$ and $w$

$$C(r) = u^2(r) + w^2(r)\ , \qquad Q(r) = u(r)w(r) - \tfrac{1}{\sqrt{8}}w^2(r)\ , \tag{1.5}$$

$j_n$ are spherical Bessel functions of order $n$, and $\tau = Qr/2$. In the NRIA, both of the $D^{SO}$ terms are zero.

In the relativistic case, the electric body form factors are

$$D_C = \Bigg(1 - \underset{\text{(DF)}}{\frac{Q^2}{8m^2}} - \underset{\text{(B)}}{\frac{Q^4}{16m^2}\frac{d}{dQ^2}}\Bigg) F_0 + \Delta_C$$

$$\frac{Q^2}{6\sqrt{2}M_d^2} D_Q = \Bigg(1 - \underset{\text{(DF)}}{\frac{Q^2}{8m^2}} - \underset{\text{(B)}}{\frac{Q^4}{16m^2}\frac{d}{dQ^2}}\Bigg) F_2 + \Delta_Q$$

$$D_C^{SO} = \frac{Q^2}{8m^2}\int_0^\infty dr\ \left[\,j_0(\tau) + j_2(\tau)\right] w^2 + \Delta_C^{SO}$$

$$\frac{Q^2}{6\sqrt{2}M_d^2} D_Q^{SO} = \frac{3}{2m^2}\int_0^\infty dr\ j_2(\tau)\ w\left[\frac{d}{dr}\left(\frac{u}{r}\right) - \frac{1}{\sqrt{2}\,r}\frac{dw}{dr}\right] + \Delta_Q^{SO} \tag{1.6}$$

where the identification and physical interpretation of the terms *depends on the method used to calculate them.*

One method (referred to as the generator, or *G-method* in this chapter), is based on the algebra satisfied by the generators of the Poincare group [see BT53, Fo61, 77, Os68, KF70, 74, FK75, CO70, 71, CC70, Li73, Fr73, 75a, CO75, CH76, and CP82]. In the language of the *G-method,* the terms labeled "DF" and "SO" arise form the Darwin Foldy and spin orbit corrections to the single nucleon current [see, for example, Fr73, and sec. 2.1 below], and those labeled "B" come from the effects of boosting the deuteron wave functions to a moving coordinate system, always necessary in a scattering process [discussed in sec. 2.2]. To obtain the additional $\Delta$ correction terms in the framework of the *G-method* requires consideration of the time ordered diagrams on which the underlying physics is presumably based. This leads to corrections due to (*a*) retardation effects due to the "recoil" graphs (in which the a meson is being exchanged at the same time the photon is interacting with one of the nucleons), together with compensating corrections from the diagonalization of the coupled $NN$ and $NN\pi$ states involved [TH73, JL75, BR75, Fr75b, 75c, GH76b, 77, HG76, and DW76, discussed in sec. 2.5], and (*b*) contributions from processes in which the off-shell nucleon is in a negative energy state when it interacts with the virtual photon, often called "pair" diagrams [KT74, Fr75b, and GH76a, discussed in sec. 2.6]. For a one pion exchange (OPE) interaction with pure *ps* ($\gamma^5$) coupling, the result for the $\Delta$ terms is

$$\Delta_C = \frac{Q^2}{m^2}\frac{d}{dQ^2}(J_0 + J_0') \qquad \Delta_Q = \frac{Q^2}{m^2}\frac{d}{dQ^2}(J_2 + J_2')$$

$$\Delta_C^{SO} = \frac{Q^2}{m^2}\frac{d}{dQ^2}J_0' \qquad \Delta_Q^{SO} = \frac{Q^2}{m^2}\frac{d}{dQ^2}J_2' \tag{1.7}$$

where the $J$'s are

$$J_0 = k_\pi \int_0^\infty dr\ j_0(\tau)\left[Y_0(x)\ C(r) + 4\sqrt{2}\ Y_2(x)\ Q(r)\right] - J_0'$$

$$J_0' = 2k_\pi \int_0^\infty dr\ j_0(\tau)\ \frac{Y_1(x)}{x}\ \left[C(r) + 4\sqrt{2}\ Q(r)\right]$$

$$J_2 = k_\pi \int_0^\infty dr\ j_2(\tau)\left[Y_0(x)\ Q(r) + \sqrt{2}\ Y_2(x)\left(C(r) - \sqrt{2}\ Q(r)\right)\right] - J_2'$$

$$J_2' = -k_\pi \int_0^\infty dr\ \frac{Y_1(x)}{x}\left[\ j_0(\tau)\left(2\sqrt{2}\ C(r) - 2\ Q(r)\right) - \frac{27}{\sqrt{2}}\ \frac{j_1(\tau)}{\tau}\ w(r)^2\right] \tag{1.8}$$

and

$$k_\pi = \frac{m_\pi^{\ 3}}{4m}\left(\frac{g_\pi^{\ 2}}{4\pi}\right)\ ,\ x = m_\pi r$$

$$Y_0(x) = \frac{e^{-x}}{x},\quad Y_1(x) = \frac{e^{-x}}{x}\left(1 + \frac{1}{x}\right)\ ,\quad Y_2(x) = \frac{e^{-x}}{x}\left(1 + \frac{3}{x} + \frac{3}{x^2}\right) \tag{1.9}$$

The same correction terms have been derived using and alternative method (referred to in this chapter as the *C-method*) based on the use of covariant Feynman diagrams in which one nucleon (the spectator) is restricted to its mass shell [Gr65, 66, CG67, and AC80]. In the *C-method*, the form factors can be expressed as integrals over the relativistic wave functions of the deuteron. In addition to the familar large $S$ and $D$ state components, there are two small $P$ state components, denoted $v_t$ and $v_s$ for spin triplet and singlet states, respectively [Re72, HG73, and BG79]. The correction terms in the *C-method* arise from only three sources: (*a*) matrix elements of the one body charge operator, discussed in sec. 2.3, (*b*) effects of boosting the deuteron wave functions, sec 2.3, and (*c*) contributions from negative energy or "pair" states, sec. 2.6. To first order in $(v/c)^2$, the corrections can be cast into the general form (1.6), but the interpretation of the terms is different. Now the corrections labeled "SO" and "DF" are *all part of the contributions arising from the boost, with the one nucleon charge operator contributing no spin independent corrections to this order*. In this method, the $\Delta$ terms are written in a form similar to (1.7):

$$\Delta_C = \frac{Q^2}{m^2}\frac{d}{d\,Q^2} I_0 \qquad\qquad \Delta_Q = \frac{Q^2}{m^2}\frac{d}{d\,Q^2} I_2$$

$$\Delta_C^{SO} = \frac{Q^2}{m^2}\frac{d}{dQ^2} I_0^{SO} \qquad\qquad \Delta_Q^{SO} = \frac{Q^2}{m^2}\frac{d}{dQ^2} I_2^{SO} \tag{1.10}$$

where the $I$'s, to first order in $(v/c)^2$, are:

$$I_0 = \int_0^\infty dr\ \ j_0(\tau)\ (u\ \hat{u} + w\ \hat{w}) \qquad I_2 = \int_0^\infty dr\ \ j_2(\tau)\left(\tfrac{1}{2}[u\ \hat{w} + w\ \hat{u}] - \tfrac{1}{\sqrt{8}} w\ \hat{w}\right)$$

$$I_0^{SO} = -2\int_0^\infty dr\ \frac{m}{r}\ j_0(\tau)\left(\tfrac{1}{\sqrt{3}}\, u\left[v_s - \sqrt{2}\, v_t\right] - \sqrt{\tfrac{2}{3}}\, w\left[v_s + \tfrac{1}{\sqrt{2}} v_t\right]\right)$$

$$I_2^{SO} = -\int_0^\infty dr\ \frac{m}{r}\left\{ j_0(\tau)\left(\sqrt{\tfrac{2}{3}}\, u\left[v_s + \tfrac{1}{\sqrt{2}} v_t\right] + \tfrac{1}{\sqrt{3}}\, w\left[v_s - \sqrt{2}\, v_t\right]\right)\right.$$
$$\left. + j_2(\tau)\ \tfrac{3}{2}\sqrt{\tfrac{2}{3}}\ w\left[\sqrt{2}\, v_s - v_t\right]\right\} \tag{1.11}$$

where $\varepsilon$ is the deuteron binding energy, and

$$\hat{u} = \left(-\frac{d^2}{dr^2} + m\varepsilon\right)u \qquad\qquad \hat{w} = \left(-\frac{d^2}{dr^2} + \frac{6}{r^2} + m\varepsilon\right)w \ . \tag{1.12}$$

Note that (1.12) are the left hand side of radial Schrodinger equations for the reduced $S$ and $D$ state wave functions, and hence can be expressed in terms of the $NN$ potentials. For this reason these correction terms have sometimes been referred to as "potential" corrections.

The derivation of these equations using both the *G*- and *C-methods* will be discussed in Part II. Both derivations were developed over a period of more than a decade, and it was not until the end of this effort that it was found that, using the same representation [in the sense of Friar (Fr77a, 80), see sec. 2.7 below] they give *identical results for OPE potentials with ps coupling* [Gr78, Fr80]. This very satisfying result gives confidence in both methods, which seem to treat the problem very differently. However, in view of the large uncertainties inherent in any attempt to extract relativistic corrections from non-relativistic calculations [see discussion in section 2.7 below], and in view of the fact that *C-method* is the only one of the two which treats all effects in a completely consistent manner, the agreement is perhaps most useful in assuring one that the *C-method* relativistic wave functions have been defined in a physically sensible way.

The numerical size of relativistic corrections to the $A$ structure function, Eq. (1.2), has become a benchmark for comparing the calculations of several different groups. Figure 1 compares the exact *C-method* result [labeled "full theory"], obtained without making the $(v/c)^2$ expansion, to (*a*) the approximate $(v/c)^2$ result obtained by inserting (1.11) into (1.6) [labeled "with potential terms"] and to (*b*) the approximate result in which only the correction terms labeled "DF" and "B" are retained in Eq. (1.6) [labeled "without potential terms" in the figure]. Note that the $\Delta$, or "potential" terms, are numerically the largest effect at small $Q^2$.

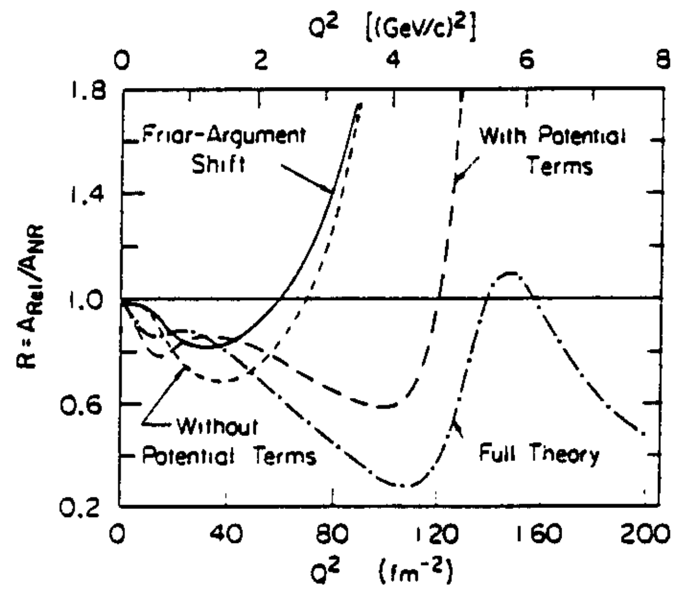


Fig. 1. *The ratio of the A structure function of the deuteron, calculated with various approximate relativistic formulas, to A calculated with the nonrelativistic formulas (1.4). All curves are for the Reid soft core wave function and dipole nucleon form factors. The Friar argument shift was a guess made in Fr73. [Figure from AC80.]*

In Fig. 2, three relativistic calculations using quite different methods are compared. One is the *C-method* described above [AGC is AC80], another is a calculation using the Bethe Salpeter equation [ZT is ZT80], and the third is a method using light front dynamics [CCKP is CC88]. It appears that these very different methods give remarkably similar results, ***when expressed as a ratio***. If nonrelativistic model calculations of the deuteron form factors are thought to be a better representation of the *NN* dynamics than existing relativistic calculations, this ratio can be used to "correct" the nonrelativistic calculations. Using this approach, Platchkov, *et. al.* [PA90] extracted an improved estimate of the neutron charge form factor, $G_{En}$, from their very preceise *ed* scattering data. Good, covariant calculations of *NN* scattering are now becomming available [see, for example GV90] and are being used in fully covariant calculations of the deuteron form factors [HT89]. As these models mature, they will replace nonrelativistic calculations and eliminate the need for making separate relativistic corrections.

From Fig. 2, the slope of the correction at $Q^2 = 0$ is about

$$\frac{\Delta A}{A} \approx -(2 \text{ or } 3)\,\frac{Q^2}{8\,m^2} \tag{1.13}$$

or two to three times the size of the Darwin Foldy correction alone, and comparable to the known slope of the neutron charge form factor at $Q^2 = 0$. Hence, on a scale defined by

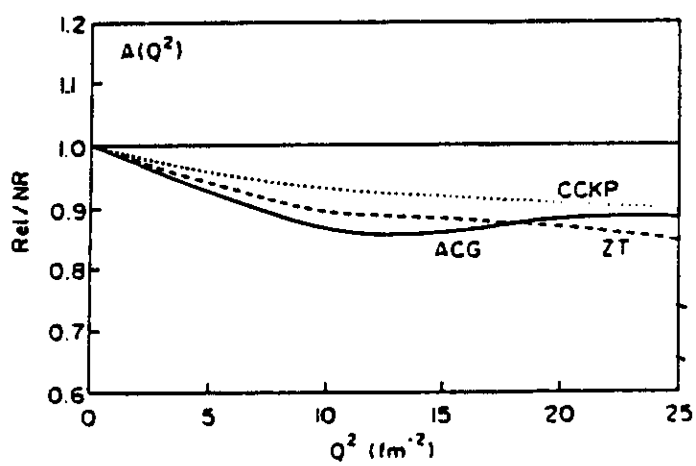


Fig. 2. *The ratio of the A structure function calculated fully relativistically to its nonrelativistic limit for three different relativistic calculations described in the text. [From PA90.]*

the neutron charge form factor this correction is important, and without it the measured size of the deuteron, proton, and neutron cannot be reconciled with $NN$ scattering data [CG67].

At larger $Q^2$, the relativistic calculations indicate that the form factors cannot be understood from the dynamics defined by the $NN$ sector alone. Figure 3 compares the theoretical results of Hummel and Tjon [HT89] with the excellent data available at high $Q^2$. Note that their curve labeled IA (which is the RIA in the language of this chapter) falls way below the data for $A$, and does not correctly reproduce the shape of $B$, in general agreement with earlier relativistic results [AC80 and ZT80]. They are able to correct this by adding contributions from the $\rho\,\pi\,\gamma$, and $\omega\,\varepsilon\,\gamma$ interaction currents, which completely dominate the high $Q^2$ results. Regardless of one's view of the reasonableness of such currents, it is clear that they are not well constrained by the $NN$ sector. Perhaps quark effects are also important in this region. Relativistic effects are extremely important to such a debate because the nonrelativistic calculations of these quanities are generally quite

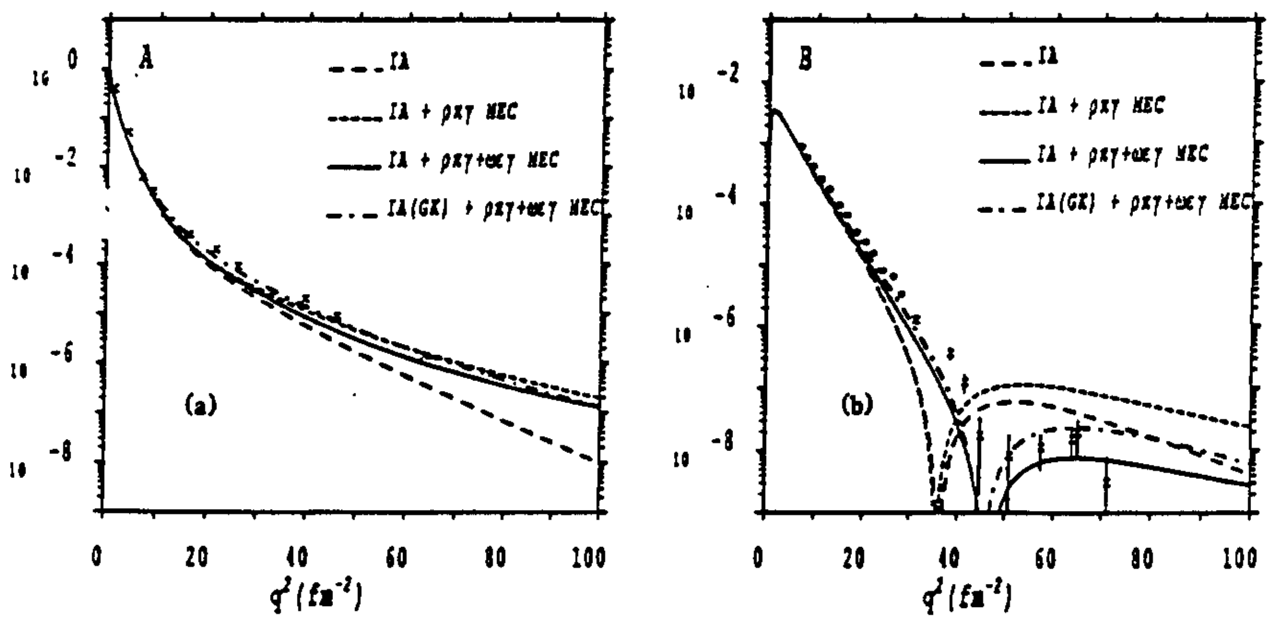


Fig. 3. *Relativistic calculation of the electric and magnetic structure functions of the deuteron. A and B. Nucleon form factors of Hohler et. al. are used except for the curves labeled GK, which use Gari-Krumpelmann form factors.*

different, and require very different additional contributions to bring theory into agreement with experiment.

Finally, note that the relativistic corrections to the quadrupole moment of the deuteron can be calculated by evaluating Eq. (1.6) at $Q^2 = 0$. A similar calculation of the magnetic form factor gives the following result, to order $(v/c)^2$, for the magnetic moment of the deuteron [Gr75, AC80]:

$$\mu_d = \mu_S\left(1 - \tfrac{3}{2} P_d\right) + \tfrac{3}{4} P_d + m\frac{1}{\sqrt{3}} \int_0^\infty r\, dr \left( u\left[\frac{1}{\sqrt{2}} v_t - v_s\right] - w\left[ v_t + \frac{1}{\sqrt{2}} v_s\right]\right) \tag{1.14}$$

where $P_d$ is deuteron $D$ state probability and $\mu_d$ is in nuclear magnetons. The correction term in (1.14) has the amusing property that it vanishes for pure OPE. Some time ago [Gr75] the correction term was estimated to be $\approx 0.015$, requireing $P_d$ to be about 5.9% in order to achieve agreement with the measured value. This estimate is comparable to that given by Friar [Fr79a], who reviews the long history of this subject.

## 1.2 Deuteron Photodisintegration

Relativistic corrections have also been found to play a significant role in deuteron photodisintegration [CM82]. Calculations of the $d(\gamma, p)n$ differential cross section for forward scattering, where the proton recoils in direction of the incident photon, are shown in Fig. 4. The nonrelativistic calculation is the solid line, and it fails to explain the data by a large margin. Corrections of two kinds were considered, and are shown. In the language of the *G-method*, including the DF and SO corrections to the one body current gives the (lower) dot-dashed line. The effect is large, most of it comming from the SO term, and the improvement is significant. However, when pair terms are included, they produce an effect almost as large and in the opposite direction. The (upper) dashed line in the figure is the effect of including the pair term corrections only, while the (faint) dotted line includes all effects. The "final" result represents some improvement over the initial nonrelativistic calculation.

The reason why relativistic corrections are so significant in this case is that forward

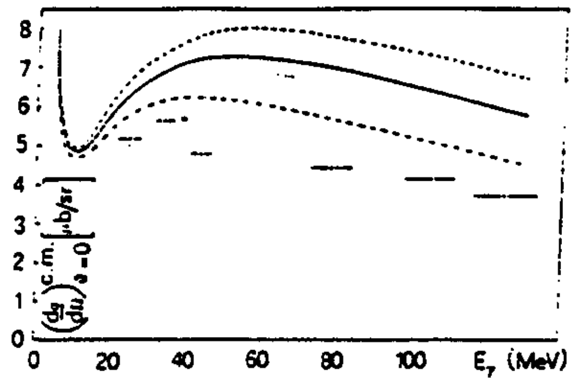


Fig. 4. *Forward photodisintegration cross section, evaluated with the Reil soft-core potential. The four curves are explained in the text. [From CM82.]*

scattering would be impossible if the deuteron were in a pure $S$ state and if the interaction had no spin dependence [CM82, Gi84]. In the absence of spin dependent effects, the nonrelativistic $E1$ transition operator, which dominates the cross section at all other angles, gives a cross section which is zero in the forward direction. The spin orbit correction to the one body operator is therefore among the major mechanisms which permits the process to occur.

## 1.3 Deuteron Electrodisintegration and Radiative Neutron Capture

Perhaps the most striking evidence for the existence of meson exchange currents (MEC) comes form the radiative capture of neutrons, $n + p \rightarrow d + \gamma$, at threshold, and from the threshold electrodisintegration of the deuteron, $e + d \rightarrow e' + (n + p)_t$, where the final $np$ state has an exicitation energy of only a few MeV. Riska and Brown [RB72], using the work of Chemtob and Rho [CR71], were the first to show that a full treatment of the MEC contributions could resolve the long standing discrepancy between the measured radiative capture cross section at threshold and theoretical calculations, which were too low by about 10%. In the language of the *G-method*, the major effects come from ($a$) the pair diagram [or, in the context of $pv$ ( $\gamma^5\gamma^\mu$ ) coupling, the seagull diagram, see Fig. 5], ($b$) interactions of the photon with the exchanged pion, and ($c$) diagrams describing the virtual

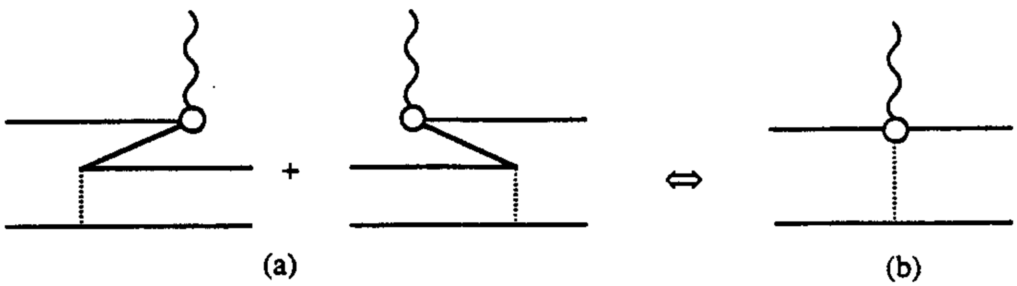


Fig. 5. *Diagramatic representation of (a) pair terms, and (b) the $\gamma\pi NN$ seagull, sometimes called a "contact" or "catastrophic" term. The strength of the seagull is fixed, through gauge invariance, to the strength of the pv coupling, while the diagrams (a) give leading order contributions only for ps coupling. In this chapter, as in the literature, the words "pair terms" are often used to refer to either diagram, because the lowest order (in $m^{-2}$) contributions from (a) in a pure ps theory equal the lowest order contributions from (b) in a pure pv theory, as indicated by the double arrow in the diagram. The time ordered Z structure of diagram (a) shows that the photon has produced a virtual N-Nbar pair; in the equivalent Feynman form this arises from contributions in which the nucleon is in a negative energy state (see Fig. 12).*

electromagnetic excitation of the $\Delta$ resonance. In a similar manner, the same mechanisms explain the $Q^2$ dependence of threshold electrodisintegration [HR73]. This success is even more striking, because the nonrelativistic impulse approximation (NRIA) has a *zero* in the amplitude which describes the transition to the $^1S_0$ final state and which is normally the dominant process. [The transition to the $^3S_1$ final state is very much suppressed at threshold, because the $^3S_1 - {}^3D_1$ scattering wave function is orthogonal to the deuteron wave function.] The MEC effects completely dominate the cross section near the minimum, filling in the "dip" and giving excellent agreement with data.

Study of sizes of the various contributions to both phenomena [GH73, HR73, LF75, and Gr76] shows that the pair (or seagull) diagram is the largest of all the effects, accounting for at least half of the total contribution in all cases, and often explaining as much as 70 – 80% of the result. Hence, in the framework of *ps* coupling, where the contribution is a true pair diagram involving negative energy states of the virtual nucleon, the largest effect is a relativistic "correction", and these processes also provide another example of a system where relativistic effects are important. They are important here for preceisely the same reason they are important in forward photodisintegration; the mechanism which normally dominates the reaction is highly suppressed. However, in the context of $pv$ coupling, the true pair terms are very small, and the same effect comes from the seagull diagram. In this case these contributions are true MEC, and relativistic effects play a smaller role.

Of course, it makes no difference what name we give to these effects as long as they are included and properly calculated! And this is the main point: to properly include all effects, without double counting or leaving things out, it is critical to (i) treat the $\pi NN$ coupling consistently, and (ii) build in the constraints implied by gauge invariance and chiral symmetry. Both symmetries relate relativistic effects and MEC to the underlying dynamics. For example, gauge invariance insures that the lowest order contributions from pair and seagull diagrams are equal, but the higher order effects are not. Pair terms will generate isoscalar contributions of order $m^{-2}$, while seagull diagrams are purely isovector, and it might even be appropriate to use the axial form factor at the $\gamma\pi NN$ vertex [LF75]. Chiral symmetry will relate the $ps$ coupling strength of the pion to $\sigma$-like $\pi\pi NN$ contact terms. Again, lowest order effects will be independent of ratio of $ps$ to $pv$ coupling strength, but higher order effects will not. The final result of a relativistic calculation will depend on whether $ps$ or $pv$ coupling (or a mixture) is used, and processes where pair (or seagull) diagrams dominate are expected to be particularly sensitive to relativistic effects.

When pair terms are dominant in the language of the *G-method*, it means in the covariant, or *C-method*, language that negative energy components are important. In the covariant language, the cross section for radiative neutron capture by the deuteron from a (pure) ${}^1S_0$ initial state is proportional to the square of the following matrix element [DG76]

$$
\begin{aligned}
M = {} & G_M \int_0^\infty dr\; u(r)\; y(r) \\
& + G_E \int_0^\infty dr \frac{1}{\sqrt{3}} mr \left\{ \left[ u(r) + \frac{1}{\sqrt{2}}\, w(r) \right] z(r) + \frac{1}{\sqrt{2}}\, y(r)\; v_t(r) \right\} \quad + \quad \Delta
\end{aligned}
\tag{1.15}
$$

The first integral in this matrix element is the usual overlap of the deuteron wave function with the ${}^1S_0$ scattering wave function, and the second integral gives the leading relativistic corrections expressed as overlaps between large components and smaller relativistic components. The first term in the second integral is an overlap between the large $S$ and $D$

state deuteron wave functions and the small relativistic component of the $^1S_0$ scattering wave function, denoted by $z$, and the second is an overlap between the large component of the $^1S_0$ scattering wave function and the small relativistic component of the deuteron (only the triplet component enters). Additional MEC terms are contained in $\Delta$. It was shown in DG76 that this expression could reproduce the nonrelativistic results, including the pair terms.

It would be worthwile to apply this method to the threshold electrodisintegration calculation. If the pion had pure *ps* coupling, the RIA, which automatically includes all pair contributions, might do very well explaining the data, restoring the correctness of the impluse approximation (but the RIA, not the NRIA). However, any realistic calculation must use wave functions determined from an interaction which fits the *NN* data, and it does not appear to be possible to do this with pure *ps* coupling [FT75]. But realistic models with pure *pv* and some admixture of *ps* coupling are being developed, and fully covariant calculations of these processes will be available soon.

Recently it has been found [MR90] that relativistic effects are also important to the analysis of $d(e, e'p)n$ coincident experiments near the quasi elastic peak. Relativistic calculations of these cross sections are especially important as a compliment to the program of such measurements planned for the new generation of electron accelerators.

## 1.4 Three Nucleon Form Factors

Relativistic effects also seem to play an important role in the form factors of the three body nuclei, $^3$He and $^3$H. The situation has come into sharp focus with the mature three body calculations of the Hannover group [SH87] and the precision measurements recently caried out at Saclay and Bates. Figure 6 shows their calculation without relativistic corrections (solid line) and including the relativistic corrections outlined below (dashed line). The relativistic corrections are large, and help bring the theory closer to experiment.

The corrections included in the Hannover calculation are those also included in the deuteron form factor calculations outlined in section 1.1 above, except that here the isovector parts of the correction terms also contribute. In the language of the *G-method*,

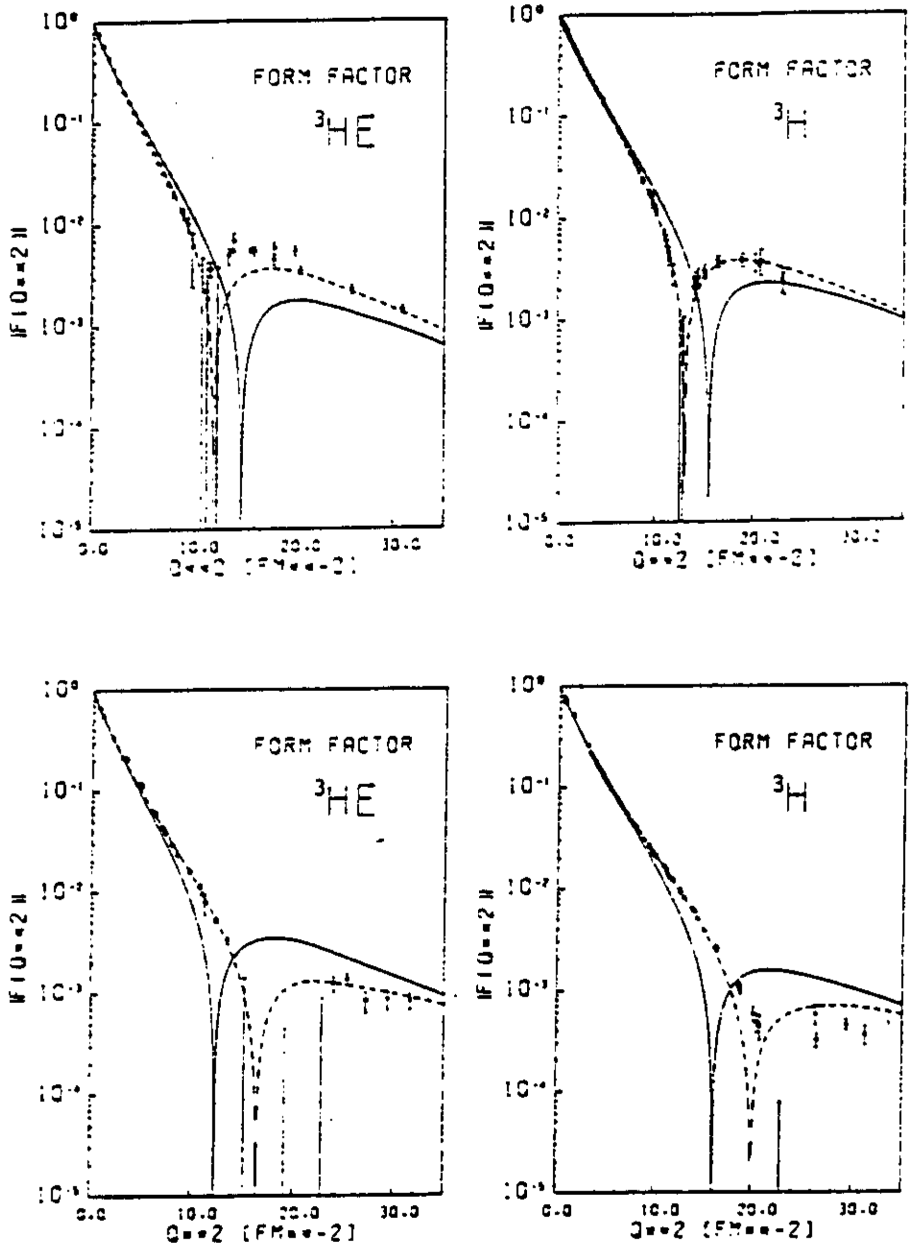


Fig. 6. *Form factors of $^3He$ and $^3H$, normalized to unity at $Q^2 = 0$. The curves are taken from SH87. The top two figures are the charge form factors; the bottom two are the magnetic. The solid line in each figure is based on their force model A2. The dashed line includes the relativistic corrections discussed in the text.*

the Hannover group includes the DF and SO corrections to the one body charge operator, which give a very small contribution, and the contribution from pair terms, which gives the bulk of the effect shown in the figure.

The precise role of three body wave functions, MEC, and relativistic effects in building up the three nucleon form factors is presently uncertain. Recent calculations [SP89], using very different MEC, give quite a good description of the form factors, and this subject will continue to be a critical testing point for few body theory. The first results from a new generation of covariant calculations are now available [RT88, Ru90], and show that relativistic effects may be important. More realistic calculations should be available soon.

The magnetic moments of $^3$H and $^3$He are also sensitive to relativistic effects [see, for example, Fr79b].

## 1.5 Nucleon-Nucleon Interaction

As a last example of the importance of relativistic effects, consider the $NN$ interaction itself. Relativistic methods for treating the $NN$ problem will be reviewed in Part III. This section will review one aspect of the problem, which is closely related to the role of pair terms discussed above.

An off-shell nucleon propagates as a superposition of positive and negative energy states. As described above, the pair term contributions arise from the coupling of these negative energy state components to the photon, and they give significant contributions to a number of processes. It is natural to expect that they might also give significant contributions to the $NN$ interaction itself, and it appears that this is indeed the case.

A simple way to describe pair processes consistently is to use a covariant equation in which one particle is off-shell and the other particle is on-shell [Gr69, 74, and 82a]. This spectator equation (sometimes referred to as the Gross equation) can be reduced to nonrelativistic form with an effective potential of the form

$$V_{eff}(r) = V^{++}(r) + \frac{\left[V^{-+}(r)\right]^{\dagger} V^{-+}(r)}{2m} \tag{1.17}$$

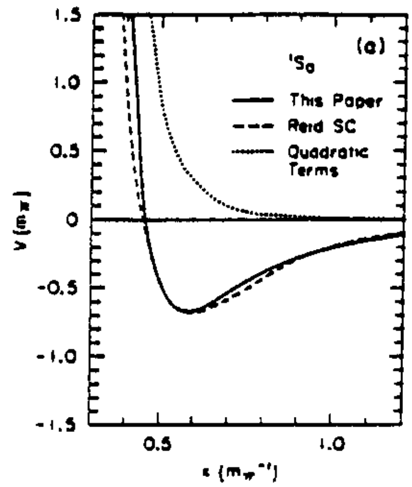


Fig. 7. *Fit to the Reid $^1S_0$ potential using a potential of the form given in Eq. (1.17). The solid line is the complete potential (1.17), the dashed line is the Reid soft core potential, and the dotted line is the quadratic term alone. [From Gr74.]*

where $V^{++}$ is the usual nonrelativistic potential constructed from $u$ spinor matrix elements of the relativistic Dirac operators describing one boson exchange (OBE), and $V^{-+}$ is the "off-diagonal" relativistic part constructed from matrix elements with the initial nucleon in a $u$ spinor positive energy state and the final nucleon in a $v$ spinor negative energy state. The second term in (1.17), which depends quadratically on the $V^{-+}$ potential, therefore describes, in lowest order, the effect of virtual pairs on the *NN* interaction.

This quadratic term plays an important role in the *NN* interaction. To get a rough estimate of its size and behavior [Gr74], a OBE model for $V$ was used, and the parameters of the model were adjusted to fit the Reid soft core potentials in all channels. Such a fit shows that these terms play an important role in the description of the short range interaction, as illustrated in Fig. 7. A complete calculation, in which the equations are solved exactly for a OBE model without making nonrelativistic approximations, and the OBE parameters determined by a fit to the *NN* data, was recently completed [GV90]. The results of this improved calculation support the conclusion that pair effects can be important in the *NN* interaction.

## 1.6 Conclusions, Discussion, and Assessment

Several conclusions can be drawn from the previous sections, and from the supporting discussion in Parts II and III (see sections 2.4, 2.7, and 3.5 below). The major conclusions from the all of this discussion are collected here:

(i) While the best evidence for relativistic effects undoubtedly comes from atomic physics [Fr79b, 81], there is ample evidence for the importance of relativistic effects in nuclear physics. These include: *(a) corrections to the charge radius of the deuteron necessary to reconcile nuclear force models with data, (b) corrections to deuteron photodisintegration in the forward direction, also necessary to reconcile theory with data, (c) contributions to radiative neutron capture, deuteron electrodisintegration, and the three nucleon form factors arising from pair (or seagull) processes, and (d) contributions to the NN interaction which are helpful in model building.*

(ii) Relativisitc effects are most likely to play an important role when (*a*) the dominant mechanisms are suppressed (as for deuteron photo- and electrodisintegration), (*b*) precision measurements are to be compared with a precise theory in the attempt to extract a small effect (as in the case of the deuteron form factors at low $Q^2$, or the charge radius and static moments), or (*c*) theory and experiment are compared at high $Q^2$ (as for the deuteron and three nucleon form factors).

(iii) Contributions from pair processes, which play a decisive role in the electrodisintegration reactions and three body form factors, and make important contributions to the *NN* interaction, *cannot be unambiguously distinguished from MEC effects. They are intertwined by the requirements of current conservation and gauge invariance, and coupled to other mechanisms by chiral symmetry. The importance of such mechanisms which are intertwined with relativity, gives a strong motivation for the development of covariant techniques closely connected to field theory.*

(iv) Using different methods, identical relativistic corrections may arise from different physical origins. *There is no unique, method independent way, of determining the size of relativistic corrections arising from the nuclear current, the boost of the nucleus, or from pair terms. However, using different methods, and summing* ***all*** *corrections consistently within each method, may give the* ***same result***.

(v) The size and nature of relativistic corrections depend on the dynamics used to describe the underlying *NN* interaction. *If this dynamics is not specified, it is not possible to determine relativistic corrections uniquely. This makes it difficult to combine calculations of relativistic effects with phenomenological nonrelativistic model calculations. The only way to insure consistency (count everything and avoid double counting) is to base the calculation on a fully relativistic approach.*

(vi) Successful, fully relativistic approaches exist for the treatment of few body systems, and are rapidly being developed for many body systems.

This concludes the overview of relativistic effects in the electromagnetic interactions of few body systems. Part II will present a detailed discussion and derivation of the results of sec. 1.1, and Part III will review the fully covariant methods.

## II. Relativistic Corrections to Order $(v/c)^2$

In this Part, lowest order relativistic corrections to nuclear electric form factors will be reviewed in some detail. Some of the results obtained will be applicable to nuclei of mass number $A$, but all detailed calculations will be given only for the deuteron form factors. Corrections come from four effects. These are (i) matrix elements of the single nucleon charge operator, (ii) boosting of the nuclear wave functions, (iii) retardation, or the dependence of the *NN* interaction on the relative time (or energy), and (iv) "pair" terms, or the coupling of the electromagnetic current to nucleons in their negative energy state. Each of these effects will be discussed separately in the following sections.

Two methods have been developed for calculating these effects. The generator method ($G$) determines the boost operators from the algebra of the Lorentz group. The covariant method ($C$) determines the transformation of the wave functions from an examination of their relativistic structure.

## 2.1 Corrections to the One Body Charge Operator

Begin by considering the electromagnetic interaction of a nucleon in the nuclear medium. The electromagnetic current operator in Dirac space is usually taken to be

$$j^{\mu} = F_1(q^2)\gamma^{\mu} + iF_2(q^2)\frac{\sigma^{\mu\nu} q_{\nu}}{2m} \tag{2.1}$$

where $F_1$ and $F_2$ are the Dirac and Pauli form factors, respectively, $\gamma^{\mu}$ and $\sigma^{\mu\nu}$ are Dirac matrices (I will use the notation of BD65), and $q^{\mu}$ is the four-momentum transfered *to* the nucleon (by the scattered electron). The form factors are normalized to $F_1(0) = e_N$ and $F_2(0) = \kappa_N$, where $e_N$ and $\kappa_N$ are the charge and anomalous magnetic moment of the nucleon. Using the isospin formalism to describe the nucleon, these quanities are operators in isospin space:

$$e_N = \tfrac{1}{2}\left(1 + \tau^3\right), \qquad \mu_N = \kappa_p\tfrac{1}{2}\left(1 + \tau^3\right) + \kappa_n\tfrac{1}{2}\left(1 - \tau^3\right) \tag{2.2}$$

If the nucleon is on shell, so that its four-momentum squared is equal to its mass squared, $p^2 = p^{\mu}p_{\mu} = m^2$, the form (2.1) can be shown to be the most general possible. For bound nucleons, which are off-shell, more general forms are possible, but (2.1) is generally used. If the nucleons are weakly bound their Dirac structure is well approximated by the positive energy spinor $u$, and the corresponding matrix element of the nuclear current is

$$\left\langle j^{\mu}\right\rangle = \left(\frac{m^2}{E(p')E(p)}\right)^{\frac{1}{2}} \bar{u}(p')\left[F_1(q^2)\gamma^{\mu} + iF_2(q^2)\frac{\sigma^{\mu\nu} q_{\nu}}{2m}\right]u(p) \tag{2.3}$$

where $E(p) = \sqrt{(m^2+\mathbf{p}^2)}$ is the energy of a particle of mass $m$. The $\sqrt{E}$ factors in (2.3) insure that the charge, $\langle j^0\rangle$, is normalized to $e_N$ at $q^2 = 0$. Expanding the matrix element in powers of the particle velocity $v^2/c^2 \approx m^{-2}$ gives, to order $m^{-2}$,

$$\langle j^0\rangle = F_1\left(1-\frac{(\mathbf{p}'-\mathbf{p})^2}{8m^2}\right) - F_2\frac{\mathbf{q}\cdot(\mathbf{p}'-\mathbf{p})}{4m^2}$$

$$+ F_1\frac{i\sigma\cdot(\mathbf{p}'\times\mathbf{p})}{4m^2} + F_2\frac{i\sigma\cdot(\mathbf{q}\times[\mathbf{p}'+\mathbf{p}])}{4m^2} \tag{2.4}$$

which holds for *arbitrary* $\mathbf{p}'$, $\mathbf{p}$, and $\mathbf{q}$. If $\mathbf{p}'-\mathbf{p}=\mathbf{q}$, then (2.4) reduces to a well known form [Fr73]

$$\langle j^0\rangle = F_1 - \left(F_1 + 2F_2\right)\left[\frac{\mathbf{q}^2}{8m^2} - \frac{i\sigma\cdot(\mathbf{q}\times\mathbf{p})}{4m^2}\right] \tag{2.5}$$

Expressed in terms of the Sachs form factors,

$$G_E = F_1 + \frac{q^2}{4m^2}F_2, \qquad G_M = F_1 + F_2 \quad , \tag{2.6}$$

this becomes

$$\langle j^0\rangle = G_E\left(1-\frac{\mathbf{q}^2}{8m^2}\right) + \left(2G_M - G_E\right)\left[\frac{i\sigma\cdot(\mathbf{q}\times\mathbf{p})}{4m^2}\right] \quad . \tag{2.7}$$

Since $G_E$ is the form factor which corresponds most closely to the charge distribution of the nucleon, the additional terms in (2.7) are interpreted as relativistic corrections. Such terms also occur in the non-relativistic reduction of the Dirac equation. The $\mathbf{q}^2/(8m^2)$ term is sometimes referred to as the Darwin-Foldy (DF) term, and originates from zitterbewegung [see BD65]. The last term is the spin-orbit (SO) term and describes the interaction of the nucleon magnetic moment with magnetic fields seen by the proton as it moves through an electric field (in this case created by the passing electron). It is this term which gives the large effect in the photodisintegration of the deuteron, described in section 1.2 above.

In making comparisons between relativistic and non-relativistic theories, it is particularly convenient to work in the Breit frame in which the electron transmits no energy to the nucleus. In this frame the three-momentum transfer is simply related to the four-

$$\frac{(\mathbf{q} \times [\mathbf{p}' + \mathbf{p}])}{4m^2} \tag{2.4}$$

n (2.4) reduces to a well known

$$\frac{(\mathbf{q} \times \mathbf{p})}{m^2}\Bigg] \tag{2.5}$$

$$2 \quad , \tag{2.6}$$

$$\frac{(\mathbf{q} \times \mathbf{p})}{4m^2}\Bigg] \quad . \tag{2.7}$$

ely to the charge distribution of as relativistic corrections. Such rac equation. The $\mathbf{q}^2/(8m^2)$ term ) term, and originates from bit (SO) term and describes the ic fields seen by the proton as it passing electron). It is this term he deuteron, described in section

d non-relativistic theories, it is i the electron transmits no energy fer is simply related to the four-

$-\mathbf{q}^2$, and we may assume that scalar functions of $\mathbf{q}^2$ generalize, scalar functions of $q^2 \equiv -Q^2$ (the SLAC convention). [See of this fact in a special case.] The non-relativistic impulse he starting point of all calculations. It is obtained [Sc64] by gnetic scalar potential of the passing electron may be treated as a time, and that the total charge operator may be taken to be the operators for each nucleon in the nucleus:

$$\left\langle J^0_{IA} \right\rangle = \sum_{\alpha=1}^{A} \left\langle j^0 \right\rangle_\alpha \, e^{i\,\mathbf{q}\cdot\,\mathbf{r}_\alpha} \tag{2.8}$$

of the $\alpha^{th}$ nucleon. Denoting the isospin operator of the $\alpha^{th}$ perator of the $\alpha^{th}$ nucleon is

$$G_{Ep}(q^2)\,\tfrac{1}{2}\left(1+\tau^3_\alpha\right) + G_{En}(q^2)\,\tfrac{1}{2}\left(1-\tau^3_\alpha\right)$$

$$G^S_E(q^2) + \tfrac{1}{2}G^V_E(q^2)\tau^3_\alpha \tag{2.9}$$

1 and neutron charge form factors, respectively.

e deuteron charge form factors will be evaluated [Ja56, Go63, rm factors can be obtained from the charge operator, which is

$$+ P_i)^0 F_D(q^2)$$

$$(P_f - P_i - q)\, 2P^0\left[\xi^*_f\cdot\xi_i\ G_C(q^2) + \left(\frac{Q^2}{6M_d^2}\right)S_{12}(\mathbf{q})\ G_Q(q^2)\right]$$

$$M_d \int d\,\mathbf{R}\,d\,\mathbf{r}\ \psi^\dagger_{P_f}(\mathbf{R},\mathbf{r}) \left\langle J^0_{IA}\right\rangle \psi_{P_i}(\mathbf{R},\mathbf{r}) \tag{2.10}$$

sor operator

$$S_{12}(\mathbf{q}) = \frac{3\,\mathbf{q}\cdot\xi_f^*\;\mathbf{q}\cdot\xi_i}{\mathbf{q}^2} - \xi_f^*\cdot\xi_i \quad , \tag{2.11}$$

$\xi_i$ and $\xi_f$ are spin-one polarization vectors of the incomming and outgoing deuteron (in their respective rest frames), and $G_C$ and $G_Q$ are the covariant monopole (sometimes called the "charge") and quadrupole form factors [see section 1.1]. In the second line of (2.10), the general form for the tensor structure of the electric matrix element has been expressed in terms of three-component vectors in the Breit frame, where $P_i = P_f = P^0$ (for a general covariant treatment, see AC80). This defines and normalizes the monopole and quadrupole form factors. The third line gives the NRIA matrix element, expressed as an integral over total and relative position coordinates of the deuteron, defined by

$$\begin{aligned} \mathbf{R} &= \tfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2) \qquad & \mathbf{r}_1 &= \mathbf{R} + \tfrac{1}{2}\mathbf{r} \\ \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 \qquad & \mathbf{r}_2 &= \mathbf{R} - \tfrac{1}{2}\mathbf{r} \end{aligned} \tag{2.12}$$

The deuteron wave functions are a product of a plain wave depending on **R**, and an internal wave function $\phi$ depending only on **r**:

$$\psi_{P_i}(\mathbf{R}, \mathbf{r}) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \phi_i(\mathbf{r})\, e^{\,i\,\mathbf{P}_i\cdot\mathbf{R} - iP^0 t} \tag{2.13}$$

Note that the time dependence of the form factor vanishes in the Briet frame. Doing the integration over **R**, and using the facts that the deuteron has isospin zero and that $\phi(\mathbf{r}) = \phi(-\mathbf{r})$, the NRIA integral for the form factor reduces to

$$F_D(q^2) = G_E^S(q^2) \int d\mathbf{r}\;\phi_f^\dagger(\mathbf{r})\; e^{\,i\,\mathbf{q}\cdot\frac{1}{2}\mathbf{r}}\,\phi_i(\mathbf{r}) \tag{2.14}$$

The non-relativistic wave function for the deuteron can be written in a convenient matrix form [BG79]:

$$\phi_i(\mathbf{r}) = \chi_2^\dagger \, \Phi_i(\mathbf{r}) \, \chi_1$$

$$\Phi_i(\mathbf{r}) = \frac{1}{\sqrt{4\pi}}\left[\frac{u(r)}{r}\,\sigma\cdot\xi_i - \frac{w(r)}{\sqrt{2}\,r}\left(\frac{3\,\mathbf{r}\cdot\,\xi_i\;\sigma\cdot\,\mathbf{r}}{r^2} - \sigma\cdot\xi_i\right)\right]\frac{i\sigma_y}{\sqrt{2}} \tag{2.15}$$

where $u$ and $w$ are the reduced $S$ and $D$ state wave functions and $\chi_1$ and $\chi_2$ are the two-component spinors of the two nucleons. Summing over nucleon spins, the form factor is quickly reduced

$$\begin{aligned} F_D(q^2) &= G_E^S(q^2)\int d\mathbf{r}\;\mathrm{tr}\left[\Phi^\dagger_f(\mathbf{r})\,\Phi_i(\mathbf{r})\right]\,e^{i\mathbf{q}\cdot\frac{1}{2}\mathbf{r}} \\ &= G_E^S(q^2)\int d\mathbf{r}\;\frac{1}{4\,\pi r^2}\left[\left(u^2 + w^2\right)\,\xi_f^*\cdot\xi_i - \sqrt{2}\left(uw - \frac{1}{\sqrt{8}}\,w^2\right)S_{12}(\mathbf{r})\right]\,e^{i\mathbf{q}\cdot\frac{1}{2}\mathbf{r}} \\ &= G_E^S(q^2)\left[\,\xi_f^*\cdot\xi_i\;F_0(q^2) + \sqrt{2}\;S_{12}(\mathbf{q})\;F_2(q^2)\right] \end{aligned} \tag{2.16}$$

with $F_0$ and $F_2$ as defined in Eq (1.4).

The DF and SO corrections to the charge operator, Eq. (2.7), modify the NRIA. Referring to the full results presented in Eq. (1.6), the DF and SO corrections give the $-Q^2/8m^2$ factors in $D_C$ and $D_Q$ [labeled DF in those equations] and the first terms in the expressions for the $D^{SO}$ form factors [eveything except the $\Delta^{SO}$'s, which come from the pair terms]. These results, in this form, were obtained by Friar [Fr73]. The DF correction is obtained by inspection from Eq. (2.7), but the SO correction requires some calculation.

The next two sections will discuss the nuclear boost effects, derived in two different ways.

## 2.2 Nuclear Boosts - Generator Method

If the nucleus is initially at rest, it will recoil when struck by the electron. Since no frame can be found in which the initial *and* final nucleus are *both* at rest, nuclear motion is always present. In the Breit frame, the nuclear motion is divided equally between the incomming and outgoing wave functions, which have momenta $-{}^1/_2\, q$ and ${}^1/_2\, q$, respectively. Nuclear motion introduces important relativistic corrections arising from Lorentz contraction and time dilation effects.

Two methods have been developed for calculating boost effects. [As it turns out, they also calculate the other effects in a different way as well!] The generator method (*G-method*), which will be discussed in this section, determines the boost operators from the algebra of the Lorentz group. A second method, the covariant method (*C-method*), determines the transformation of the wave functions from an examination of their relativistic structure, and will be discussed in the following section.

The inhomogeneous Lorentz group, or Poincare group, is described by 10 generators. There are three momentum operators $P_i$, three angular momentum operators $J_i$, three boost operators $K_i$, and the hamiltonian H, which satisfy the following algebra:

$$
\begin{array}{llll}
\left[P_i, P_j\right] = 0 & \left[P_i, H\right] = 0 & \left[J_i, H\right] = 0 & \text{(a)} \\
\left[J_i, J_j\right] = i\varepsilon_{ijk} J_k & \left[J_i, P_j\right] = i\varepsilon_{ijk} P_k & \left[J_i, K_j\right] = i\varepsilon_{ijk} K_k & \text{(b)} \\
\left[K_i, P_j\right] = iH\delta_{ij} & \left[K_i, K_j\right] = -i\varepsilon_{ijk} J_k & \left[K_i, H\right] = i P_i & \text{(c)}
\end{array}
\tag{2.17}
$$

The Galilian group also has 10 generators, which satisfy the same algebra *except* for the first two relations in (2.17c), which become instead

$$
\left[K_i, P_j\right] = i\, m\, \delta_{ij} \qquad \left[K_i, K_j\right] = 0 \tag{2.18}
$$

These algebraic relations show that the generators are not all independent of each other. The central idea behind the generator approach is to use the Poincare algebra to express the boost operator in terms if the dynamical coordinates and interactions of the system, and then to calculate the recoil effects from this operator.

A "solution" of the equations (2.17) which expresses the generators in terms of momenta, $\mathbf{p}_\alpha$, coordinates, $\mathbf{r}_\alpha$, and spins, $\mathbf{s}_\alpha$, of the $\alpha^{th}$ particle was found by Bakamjian and Thomas [BT53]. For a single particle $\alpha$, the solution is

$$
\begin{aligned}
&\mathbf{p}_\alpha \\
&\mathbf{j}_\alpha = \mathbf{s}_\alpha + (\mathbf{r}_\alpha \times \mathbf{p}_\alpha) \\
&\mathrm{h}_\alpha = \sqrt{m_\alpha^2 + \mathbf{p}_\alpha^2} \\
&\mathbf{k}_\alpha = \tfrac{1}{2}\{\mathbf{r}_\alpha , \mathrm{h}_\alpha\} - \frac{\mathbf{s}_\alpha \times \mathbf{p}_\alpha}{m_\alpha + \mathrm{h}_\alpha} - t\,\mathbf{p}_\alpha
\end{aligned}
\tag{2.19}
$$

where $m_\alpha$ is the mass, and

$$
\left[\mathrm{r}_{i\alpha} , \mathrm{p}_{j\beta}\right] = i\delta_{ij}\delta_{\alpha\beta} \qquad \left[\mathrm{s}_{i\alpha} , \mathrm{s}_{j\beta}\right] = i\varepsilon_{ijk}\,\mathrm{s}_{k\alpha}\,\delta_{\alpha\beta}
\tag{2.20}
$$

For $A$ non-interacting particles, the generators are the sum of the independent generators (2.19).

When interactions are included, it is convenient to introduce collective coordinates. The older literature contains considerable discussion of methods for defining collective coordinates (relativistic center of mass) and for introducing interactions into the generators [see BT53, Fo61, 77, Os68, KF70, 74, FK75, CO70, 71, CC70, Li73, Fr73, 75a, CO75, CH76, and GM81]. A major problem is to find generators which satisfy the cluster separability property, which states that any subset of the $A$ particle system, when removed to infinity, should satisfy the same dynamics that that subset would if it were interacting in isolation (except for the requirements of energy momentum conservation). This interesting problem has been solved [CP82], but will not be discussed here. The following discussion parallels the work of Friar [Fr75a], which in turn is based on the work of Osborn [Os68] and Krajcik and Foldy [KF74].

Introducing the following collective coordinates

$$
\begin{aligned}
&\mathbf{R} = \left(\frac{1}{M}\right)\sum_{\alpha=1}^{A} m_\alpha \mathbf{r}_\alpha \qquad &&\boldsymbol{\rho}_\alpha = \mathbf{r}_\alpha - \mathbf{R} \\
&\mathbf{P} = \sum_{\alpha=1}^{A} \mathbf{p}_\alpha &&\boldsymbol{\pi}_\alpha = \mathbf{p}_\alpha - \frac{m_\alpha}{M}\mathbf{P}
\end{aligned}
\tag{2.21}
$$

where $M = \sum m_\alpha$, the generators can be written

$$\mathbf{P} = \sum_{\alpha=1}^{A} \mathbf{p}_\alpha$$

$$\mathbf{J} = \sum_{\alpha=1}^{A} \mathbf{j}_\alpha = \mathbf{S} + (\mathbf{R} \times \mathbf{P}) \qquad \text{where} \quad \mathbf{S} = \sum_{\alpha=1}^{A} \{(\rho_\alpha \times \pi_\alpha) + \mathbf{s}_\alpha\}$$

$$\mathrm{H} = \sum_{\alpha=1}^{A} \mathrm{h}_\alpha + U = M + \sum_{\alpha=1}^{A} \frac{\mathbf{p}_\alpha^2}{2m_\alpha^2} + U$$

$$\mathbf{K} = \sum_{\alpha=1}^{A} \mathbf{k}_\alpha + \mathbf{V} = M\mathbf{R} - t\mathbf{P} + \sum_{\alpha=1}^{A} \frac{1}{4m_\alpha} \{(\mathbf{p}_\alpha^2 \mathbf{r}_\alpha + \mathbf{r}_\alpha \mathbf{p}_\alpha^2) - 2\,\mathbf{s}_\alpha \times \mathbf{p}_\alpha\} + \mathbf{V} \tag{2.22}$$

The momentum and angular momentum operators are the same for the Galilean group, but the hamiltonian and boost operators change their form in the nonrelativistic limit, which is also given to order $m^{-2}$ in Eq. (2.22). The interactions are contained in the scalar term $U$ and the vector term **V** which appear in the total hamiltonian H and the total boost **K**. The generators (2.22) satisfy the commutation relations (2.17) provided $U$ is a scalar independent of the total coordinate **R**, and that the interaction boost is a vector under rotations, related to $U$ through [KF74]

$$[\mathrm{V}_i , \mathrm{P}_j] = i\delta_{ij} U \tag{2.23}$$

In addition, the components of **V** must satisfy the following conditions

$$\left[\mathrm{V}_i , \sum_{\alpha=1}^{A} \mathrm{k}_{\alpha\, j}\right] - \left[\mathrm{V}_j , \sum_{\alpha=1}^{A} \mathrm{k}_{\alpha\, i}\right] + \left[\mathrm{V}_i , \mathrm{V}_j\right] = 0 \tag{2.24}$$

The remaining task is to solve equations (2.23) and (2.24) for **V**, and evaluate the recoil corrections implied by the solution.

The problem is solved to order $m^{-2}$ by expanding the hamiltonian and boost operators and solving the equations systematically to this order. To this end, first examine the lowest order problem, which is equivalent to treating the Galilean invariance exactly. For this case the generators are very simple,

$$\begin{aligned} \mathrm{H}_0 &= M \\ \mathbf{K}_0 &= M\,\mathbf{R} - t\,\mathbf{P} \end{aligned} \tag{2.25}$$

and it is easy to see that, along with the **P** and **J** given in (2.22), they satisfy the algebra of the Galilean group. Using the identity $e^{a+b} = e^a e^b e^{-\frac{1}{2}[a,\,b]}$, which holds provided the commutator $[a,b]$ commutes with both $a$ and $b$, one obtains

$$\begin{aligned} e^{i\theta\cdot\mathbf{K}_0}\, e^{-iMt}|0> &= e^{iM\theta\cdot\mathbf{R} - i\left(M + \frac{\theta^2}{2}M\right)t}|0> \\ &= e^{i\mathbf{P}\cdot\mathbf{R} - i\left(M + \frac{\mathbf{P}^2}{2M}\right)t}|0> \end{aligned} \tag{2.26}$$

where $\theta = \mathbf{P}/M$ is the "velocity" the boost imparts to the state (**P** is used to denote both an operator and a $c$ number). Note that (2.26) gives the result expected for a non-relativistic state of momentum **P**.

Next, consider relativistic effects to order $m^{-2}$. Note that the choice

$$\mathbf{V} = \tfrac{1}{2}(\mathbf{R}U + U\,\mathbf{R}) \tag{2.27}$$

is consistent with (2.23) if the approximate form of **K** given in (2.22) is used, and also satisfies (2.24) to lowest order in $m^{-2}$. [Unfortunately, this choice is not unique; see the discussion in sec. 2.7 below.] To evaluate the boost, use the following identity

$$e^{a+b} = \left(1 + b + \tfrac{1}{2}[a\,,b] + \tfrac{1}{3!}[a\,,\,[a\,,\,b]] + \tfrac{1}{4!}[a\,,[a\,,[a\,,b]]]\right) e^a \tag{2.28}$$

which is valid *to first order in b*, provided the four-fold commutator $[a,[a,[a,b]]] = 0$. Taking $a = i\,\theta\cdot\mathbf{K}_0$, and $b = i\,\theta\cdot\Delta\mathbf{K}$, where $\Delta\mathbf{K}$ is the extra term of order $m^{-2}$ obtained from the approximate form of **K** given in (2.22), using (2.27), and working out all of the commutators, gives the following expression for the boosted wave function, valid to order $m^{-2}$:

$$e^{i\,\theta\cdot(\mathbf{K}_0+\Delta\mathbf{K})}\,e^{iMt}\,|0\rangle = \left(1 - i\chi - i\chi_R\right)\,e^{iM\,\theta\cdot\mathbf{R} - iM\left(1+\frac{1}{2}\theta^2\right)t}\,|0\rangle \tag{2.29}$$

with

$$\chi = -\frac{1}{2}\sum\left(\frac{\pi_\alpha^2}{2m_\alpha} + \frac{1}{2}\pi_\alpha\cdot\theta\right)\rho_\alpha\cdot\theta + \text{hc} \;+\; \frac{1}{2}\sum\frac{(\mathrm{s}_\alpha \times \pi_\alpha)\cdot\theta}{\mathrm{m}_\alpha}$$

$$\chi_R = -\left(\mathbf{R}\cdot\theta - \frac{1}{2}t\,\theta^2\right)\left[\sum\frac{\pi_\alpha^2}{2m_\alpha} + U\right] - \frac{1}{6}\,M\theta^2\,\mathbf{R}\cdot\theta + \frac{1}{24}t\,M\,\theta^4 + \frac{i}{4}\,\theta^2 \tag{2.30}$$

To obtain (2.30), the operator $\mathbf{P}$ has been eliminated by working it to the right and replacing it by its eigenvalue $M\theta$. Next, observe that the first term in $\chi_R$ can be further reduced by allowing the term in square brackets to operate directly on the internal coordinates of the state $|0\rangle$, which gives (minus) the binding energy of the state $M_B - M$. This term corrects the mass in the exponential, replacing $M$ by the correct bound state mass $M_B$ in the lowest order terms. Note that the last term changes the non-relativistic norm of the state. The total effect of $\chi_R$ is therefore

$$\left(1 - i\chi_R\right)\,e^{iM\theta\cdot\mathbf{R} - iM\left(1+\frac{1}{2}\theta^2\right)} = \left(1 + \frac{1}{4}\theta^2\right)\,e^{iM_B\theta\cdot\mathbf{R}\left(1+\frac{1}{6}\theta^2\right) - iM_B t\left(1+\frac{1}{2}\theta^2+\frac{1}{24}\theta^4\right)} \tag{2.31}$$

To interpret these corrections, recall that the magnitude of the relativistic boost angle is related to the magnitude of the particle velocity by

$$\theta = \tanh^{-1}(v) = v + \frac{1}{3}v^3$$

and hence substituting $v$ for $\theta$ in the exponent of (2.31) gives the expansions, to order $m^{-2}$, of the correct relativistic bound state energy and momentum. This permits $\chi_R$ to be removed from (2.29) and gives [Os68, Fr75a]

$$e^{i\,\theta\cdot(\mathbf{K}_\circ+\Delta\mathbf{K})}\,e^{iMt}\,|0> = (1-i\chi)\left(\frac{E_B(P_B)}{M_B}\right)^{\frac{1}{2}} e^{i\,\mathbf{P}_B\cdot\mathbf{R}-iE_B(P_B)t}\,|0> \tag{2.32}$$

with $\theta = \mathbf{P}_B/M_B$ in $\chi$. The factor of $\sqrt{E_B}$ on the right hand side of (2.32) is expected; it changes the non-relativistic normalization, but preserves the covariant relativistic norm, with volume element

$$\int d^4P_B\;\delta\left(M_B^2 - P_B^2\right) = \int \frac{d\mathbf{P}_B}{2E_B}\quad . \tag{2.33}$$

To obtain the boost corrections implied by (2.32), note that the nuclear charge operator (2.8) can be modified, to order $m^{-2}$, to include the extra factors in (2.32)

$$\left(\frac{E_B(P_f)E_B(P_i)}{M_B^2}\right)^{\frac{1}{2}} \Psi^\dagger_{P_f}\left(1+i\chi(P_f)\right)\left\langle \rho_{IA}\right\rangle\left(1-i\chi(P_i)\right)\Psi_{P_i} = \frac{E_B(\frac{1}{4}q^2)}{M_B}\;\Psi^\dagger_{\frac{1}{2}\mathbf{q}}\left\langle \rho_{IA}\right\rangle\Psi_{-\frac{1}{2}\mathbf{q}}$$
$$+\;\Psi^\dagger_{\frac{1}{2}\mathbf{q}}\;\left[\chi(\tfrac{1}{2}\mathbf{q})\left\langle \rho_{IA}\right\rangle - \left\langle \rho_{IA}\right\rangle\chi(-\tfrac{1}{2}\mathbf{q})\right]\Psi_{-\frac{1}{2}\mathbf{q}} \tag{2.34}$$

where the right hand side has been specialized to the Breit frame.

Now apply (2.34) to the deuteron. The energy factor, $E_B$, is exactly the same as the Breit frame energy factor $P^0$, so this factor cancels, and the first term on the right hand side gives the one body corrections discussed in sec. 1.1. The second (boost) term can be simplified using

$$\begin{aligned}\rho_1 &= -\rho_2 = \tfrac{1}{2}\mathbf{r}\\ \pi_1 &= -\pi_2 = -\,i\nabla_{\mathbf{r}}\end{aligned} \tag{2.35}$$

so that terms in $\chi$ with an odd power of these variables sum to zero. Finally,

$$\chi = \frac{1}{16m^2}(\mathbf{q}\cdot\nabla\,\mathbf{q}\cdot\mathbf{r} + \mathbf{q}\cdot\mathbf{r}\;\mathbf{q}\cdot\nabla) \tag{2.36}$$

and a simple evaluation of this term gives the boost corrections [Fr73] to the electric form factors. They are the terms in Eq. (1.6) labeled with a "B".

The covariant method for calculating these corrections is developed in the next section.

## 2.3 Nuclear Boost - Covariant Method

An alternative way to derive relativistic corrections is to express form factors (or other matrix elements) in terms of relativistic wave functions, and then to expand these wave functions in a power series in $(q/m)^2$, where $\mathbf{q}$ is the momentum of the recoiling system (the *C-method*). The first term in this series is then the wave function in the rest system, and it is assumed that this can be approximated by a non-relativistic wave function. The next term gives the boost correction (to order $m^{-2}$) in terms of this non-relativistic wave function. However, once the amplitudes and wave functions have been defined, there is nothing in the method which requires the expansion in powers of $m^{-2}$ and with modern computers it is perhaps just as easy to evaluate the matrix elements without making such an expansion, giving results valid to *all* orders in $m^{-2}$. Early work [see Gr65, 66, CG67] calculated the correction terms only to first order in $m^{-2}$, but later [AC80] corrections were calculated to all orders. This technique has now become one of the fully covariant methods which will be discussed in Part III.

Related work using the Bethe-Salpeter equation to derive $(v/c)^2$ expansions can be found in Refs. BP68, and 69, and similar work using the Blankenbecler-Suger equation is

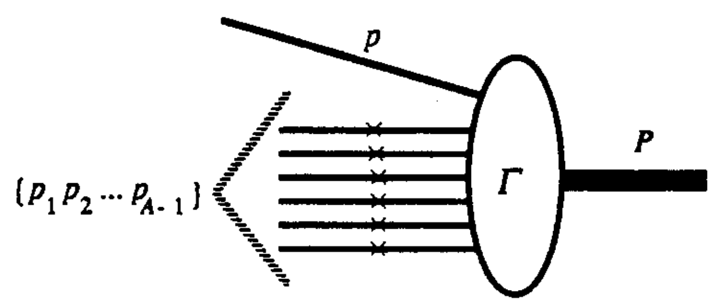


Fig. 8. *Diagramatic representation of the relativistic vertex function $\Gamma$ described in the text. The nucleus consists of A constituents, with A–1 of them on-shell (denoted by the × on the nucleon line in the diagram). The momenta are labeled.*

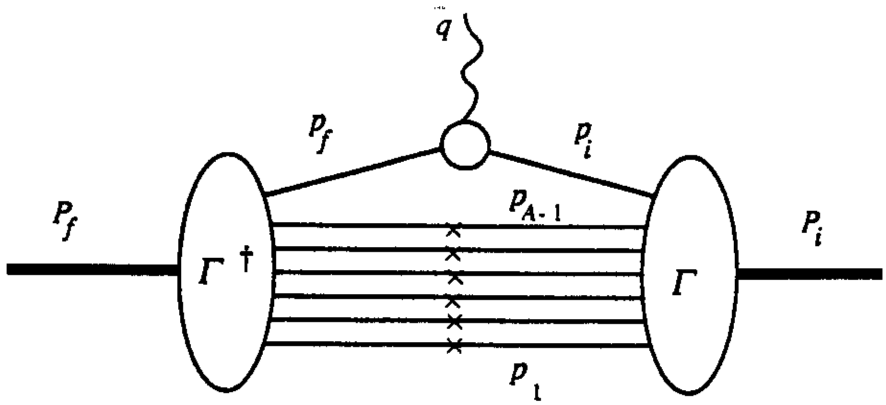


Fig. 9. *Diagramatic representation of the RIA integral (2.39) for the nuclear form factor. The A−1 spectator nucleons are on shell, and the virtual photon interacts with the off-shell nucleon. There are A diagrams of this type, one for each constituent.*

in JW84. For a nice discussion of the comparison of the *G*- and *C-methods*, in the context of field theory, see GM81.

Expanding on ideas previously published, the relativistic vertex function of an $A$ body system, from which the relativistic wave function is constructed, is defined as shown in Fig. 8. The resulting covariant matrix element for the nuclear form factor, in the relativistic impulse approximation (RIA), is shown diagramatically in Fig. 9. In the figures, the oval represents the covariant vertex function describing the coupling of the bound state to $A-1$ on-shell nucleons and one off-shell nucleon. Energy and momenta are conserved at each vertex, so if the four-momentum of the $n^{th}$ physical nucleon is $p_n$, and the four-momentum of the bound state is $P$, then the four momentum of the off shell nucleon is

$$p = P - \sum_{n=1}^{A-1} p_n \tag{2.37}$$

and it is easy to see that $p^2 < m^2$ if the system is bound. Denoting the vertex function by $\Gamma$, the relativistic wave function is

$$
\begin{aligned}
\Psi_{\lambda_n \alpha}(p_n, P) &= \Psi_{\lambda_1 \lambda_2 \ldots \lambda_{A-1}, \alpha}(p_1\, p_2 \cdots p_{A-1}, P) \\
&= \frac{N}{\sqrt{2P^0}} S_{\alpha\alpha'}(p)\, \bar{u}(p_1) \bar{u}(p_2) \ldots \bar{u}(p_{A-1}) \Gamma^{\mu}_{\alpha'}(p_1\, p_2 \cdots p_{A-1}, P)\, \xi_\mu \\
&= \frac{N}{\sqrt{2P^0}} S_{\alpha\alpha'}(p)\, \Gamma_{\alpha'}(p_n, P)
\end{aligned}
\tag{2.38}
$$

where $S$ is the propagator of an off shell nucleon, $\xi$ is the polarization vector of the nucleus (assumed to be spin-one with an eye to applications to the deuteron), and the relativistic normalization factor for an integral spin particle, $\sqrt{2P^0}$, has been added so that the normalization of $\Psi$ agrees with the non-relativistic convention. An additional normalization constant, $N$, has been added for convenience, and will be chosen later. The structure of $\Gamma$ may be very complicated, but only two facts about it are important for the coming discussion:

(i) it is manifestly covariant and its transformation under boosts is known;
(ii) in its rest frame it depends only on the three-momenta, $\mathbf{p}_n$ , and spins, $\lambda_n$ , of its on-shell constituents, the spin of the overall nuclear state, and on the *energy state* (+ or –) *of its off shell nucleon.* Only in this latter way does it differ in structure from a non-relativistic wave function.

These two features of the wavefunction will be discussed further below after the form factor has been defined.

Using this wave function the nuclear form factor, in the RIA illustrated in Fig. 9, can be written

$$
\left(P_f + P_i\right)^{\mu} F_A(q^2) = A \frac{\sqrt{4 P_f^0 P_i^0}}{N^2} \int \prod_{n=1}^{A-1} \left( \frac{d^3 p_n\, m}{(2\pi)^3\, E(p_n)} \right)
\sum_{\alpha \alpha' \lambda_n} \left\{ \bar{\Psi}_{\lambda_n \alpha'}(p_n, P_f)\, j^{\mu}_{\alpha' \alpha}\, \Psi_{\lambda_n \alpha}(p_n, P_i) \right\}
\tag{2.39}
$$

where the factor of $(P_f + P_i)$ on the left hand side is the correct relativistic tensor structure for the electromagnetic interaction of a spin zero system [BD65], and is also convenient for treating the electric interactions of a spin one system (for nuclei with other spins, appropriate factors should be used to define $F$, but the right hand side of (2.39) would be unchanged.) The non-covariant normalization factors introduced in (2.38) are cancelled by identical factors multiplying the integral on the right hand side of (2.39), and hence it is covariant. The factor $A$ counts the $A$ diagrams of Fig. 9 which contribute to the form factor, which are identical for nuclei with an equal number of neutrons and protons.

Eq. (2.39) can be obtained in two steps:

(i) use the Feynman rules to construct a diagram like that shown in Fig. 9, but with *all* internal particles off-mass shell. In addition to the integrations given above, this will involve integrations over the virtual energies of the particles $p_n{}^0$;
(ii) eliminate the internal energy integrations by using the residue theorm, but *keep the positive energy poles of the spectator particles* (those $A-1$ particles not interacting with the virtual photon) *only*.

The last step involves the replacement

$$\int \frac{dp_n^0}{2\pi} \, \frac{m + \not{p}_n}{m^2 - p_n^2} \Rightarrow \left(\frac{m}{E\,(p_n)}\right) \sum_{\lambda_*} u\,(p_n, \lambda_n)\, \bar{u}\,(p_n, \lambda_n) \tag{2.40}$$

which accounts for the factors of $m/E$ in the integrations in (2.39) and the natural appearance of the free spinors $u$ in the matrix elements in (2.38).

The original physical justification for this method is given in Gr65, and details of its application to the deuteron have been reported in Gr66, CG67, and AC80. In Gr65 in was demonstrated that, for loosely bound systems, the Feynman integral is dominated by contributions from the region where the spectator (only one for the deuteron) is close to its mass shell, and that this region is well approximated by the replacement (2.40). In addition, the replacement (2.40) is a natural covariant extension of the physical idea that interactions with a loosely bound system are dominated by the one body terms (*i. e.*

meson exchange effects are small), which is also the physics used to justify the original NRIA. Extension of this treatment to $A > 2$ body systems does not appear to have been given previously in the literature.

The next task is to use the covariance of the wave function, and its decomposition into + and – channels, to reduce (2.39) and find the boost corrections. The covariance will be discussed first. Using the representations of the Lorentz group on the Dirac spinor space [BD65] the free spinors can be shown to transform as follows [see, for example, AC80]:

$$S\,[B_i]\ u(p_n^{[i]},\lambda_n) = u(p_n,\lambda_n')\ D^{(\frac{1}{2})}_{\lambda_n'\lambda_n}(R_i)$$
$$\bar{u}(p_n^{[i]},\lambda_n)\ S^{-1}[B_i] = \bar{u}(p_n,\lambda_n')\ D^{(\frac{1}{2})\dagger}_{\lambda_n\lambda_n'}(R_i) \qquad (2.41)$$

where $S[B_i]$ [not to be confused with the propagator used in Eq. (2.38)] is the Dirac representation of the boost operator, $B_i$, which carries the four-momentum of the initial nucleus from rest, $P_R = (M_R\,,\mathbf{0})$, to $P_i$, and $D$ is the spin $^1/_2$ rotation matrix which describes the (Wigner) rotation, $R_i$, of the nuclear spin under the boost. The vectors $p_n{}^{[i]}$ are $p_n$ in the rest system, *i. e.*

$$B_i\ p_n^{[i]} = p_n \qquad (2.42)$$

Hence, the relativistic wave function (2.38) is boosted from its rest frame to the moving frame by the boost operator, $B_i$, as follows [AC80]:

$$\Psi_{\lambda,\alpha}(p_n,P_i) = \sqrt{\frac{M_R}{P_i^0}}\,S_{\alpha\alpha'}[B_{P_i}]\ \Psi_{\lambda',\alpha'}(p_n^{[i]},M_R)\ \ D^{(\frac{1}{2})}_{\lambda_n\lambda_n'}(R_i) \qquad (2.43)$$

where the Wigner rotation operator in (2.43) is a shorthand notation for a direct product of $A-1$ $D$'s , one for each spectator.

The next step is to decompose the wave function into the positive and negative energy parts referred to above. This follows from the decomposition of the propagator of an off-shell nucleon:

$$S_{\alpha\alpha'}(p) = \frac{(m + \not{p})_{\alpha\alpha'}}{m^2 - p^2 - i\varepsilon}$$

$$= \left(\frac{m}{E(p)}\right) \sum_{\lambda} \left[ \frac{u_\alpha(p,\lambda)\bar{u}_{\alpha'}(p,\lambda)}{E(p) - p^0 - i\varepsilon} - \frac{v_\alpha(-p,\lambda)\,\bar{v}_{\alpha'}(-p,\lambda)}{E(p) + p^0 - i\varepsilon} \right] \tag{2.44}$$

where the sum over $u$ spinors describes the propagation of the nucleon in its positive energy state and the $v$ spinor sum describes the propagation in its negative energy state. Using this identity, the relativistic wave function can be written in a two channel form:

$$\Psi_{\lambda_n,\alpha}(p_n, P) = \left( \prod_{n=1}^{A-1} \sqrt{\frac{E(p_n)}{m}} \right) \sqrt{\frac{m}{E(p)}} \times$$

$$\sum_{\lambda} \left\{ u_\alpha(p,\lambda)\Psi^{+}_{\lambda_n\lambda}(p_n, P) + v_\alpha(-p,\lambda)\Psi^{-}_{\lambda_n\lambda}(p_n, P) \right\} \tag{2.45}$$

where relativistic normalization factors $\sqrt{m/E}$ associated with both the on-shell nucleons and the virtual nucleon have been included in the definition of $\Psi^+$ and $\Psi^-$. Since such factors are not present in the total $\Psi$, they must be cancelled, and this is the origin of the $\sqrt{\ }$ factors in the expansion. [The additional factor of $m/E$ from (2.44) accounts for the unsymmetric appearance of these factors in (2.45). Note that, for the deuteron, all these energy factors cancel if the decomposition is done in the rest frame; in agreement with [AC80].] With this definition, $\Psi^+$ now has the structure of a non-relativistic wave function, but the $\Psi^-$ component is purely relativistic and will be discussed further in section 2.6 and Part III.

Using (2.43) and (2.45) to reduce the form factor (2.39) in the Breit frame, gives

$$F_A(q^2) = A \int \prod_{n=1}^{A-1} \left( \frac{d^3 p_n\, m}{(2\pi)^3\, E(p_n)} \right) \left( \frac{M_R}{P^0} \right) \frac{1}{N^2}$$

$$\left\{ \bar{\Psi}_{\lambda_n''}\left( p_n^{[f]}, P_R \right) j_R^0\, \Psi_{\lambda_n'}\left( p_n^{[i]}, P_R \right) D^{(\frac{1}{2})}_{\lambda_n''\lambda_n'}\left( B_f^{-1} B_i \right) \right\}$$

$$= A \int \prod_{n=1}^{A-1} \left( \frac{d^3 p_n \sqrt{E(p_n^{[f]})\, E(p_n^{[i]})}}{(2\pi)^3\, E(p_n)} \right) \left( \frac{M_R}{P^0} \right) \frac{1}{N^2}$$

$$\left\{ \bar{\Psi}^{+}_{\lambda_n'}\left(p_n^{[f]}, P_R\right) \left\langle j^0_R \right\rangle \Psi^{+}_{\lambda_n}\left(p_n^{[i]}, P_R\right) D^{(\frac{1}{2})}_{\lambda_n'' \lambda_n'}\left(B_f^{-1} B_i\right) \right\} \tag{2.46}$$

where Dirac indicies have been supressed, and the decomposition (2.45) was used in the rest frame of each nucleus. The transformed one body charge operator is

$$\begin{aligned} j^0_R &= S\,[B_f]\, j^0 S^{-1}[B_i] \\ &= F_1 \gamma^0 - F_2 \frac{Q^2}{8m^2} + F_2 \frac{\alpha \cdot \mathbf{q}}{2m} \end{aligned} \tag{2.47}$$

with

$$\left\langle j^{\mu}_R \right\rangle = \left( \frac{m^2}{E\left(p^{[f]}\right) E\left(p^{[i]}\right)} \right)^{\frac{1}{2}} \bar{u}\left(p^{[f]}\right)\, j^0_R\, u\left(p^{[i]}\right) \tag{2.48}$$

The relativistic corrections can now be evaluated. Note that the arguments of both the initial and final wave functions can be related to the integration variables through the Lorentz transfomation $B_i$, which, in the Breit frame is a boost in the direction of $-\mathbf{q}$ with boost angle $\theta = \tanh^{-1}[Q/2M_R]$:

$$\begin{aligned} p_n^{[i]} &= B_i^{-1}\, p_n \\ p_n^{[f]} &= B_f^{-1}\, p_n = B_i\, p_n \end{aligned} \tag{2.49}$$

Because the spectator particles are on-shell, it is sufficient to give the three-components of the momenta, which can be determined uniquely

$$\mathbf{p}_n^{[i]} = \mathbf{p}_n - \frac{m}{M}\mathbf{P}_i + \mathbf{P}_i\left(\frac{\mathbf{P}_i\cdot\mathbf{p}_n}{2M^2} - \frac{\mathbf{p}_n^2}{2mM} - \frac{m\varepsilon}{M^2}\right)$$

$$= \pi_n^{[i]} - \mathbf{P}_i\left(\frac{\mathbf{P}_i\cdot\pi_n^{[i]}}{2M^2} + \frac{\left(\pi_n^{[i]}\right)^2}{2mM} + \frac{m\varepsilon}{M^2}\right)$$

$$= \pi_n^{[i]} + \frac{\mathbf{q}}{2}\left(\frac{-\mathbf{q}\cdot\pi_n^{[i]}}{4M^2} + \frac{\left(\pi_n^{[i]}\right)^2}{2mM} + \frac{m\varepsilon}{M^2}\right) \qquad (2.50)$$

where $\pi^{[i]}$ are the same momenta defined in Eq. (2.21) and the binding energy $\varepsilon = M-M_R$ appears because of the difference between $M$, which appears in the definition of $\pi$, and $M_R$, which occurs in the transformation. The $\mathbf{p}^{[f]}$ are obtained from (2.50) by changing $\mathbf{q}$ to $-\mathbf{q}$. Using the transformation (2.50), the wave functions can be expanded in Taylor series to compute the corrections to lowest order. The result for the incomming state is, suppressing the superscript $[i]$ on the $\pi$ variables,

$$\Psi^+(\mathbf{p}_n^{[i]},0) \approx \left[1 + \sum_{n=1}^{A-1}\left(\frac{-\mathbf{q}\cdot\pi_n}{4M^2} + \frac{\pi_n^2}{2mM} + \frac{m\varepsilon}{M^2}\right)\frac{\mathbf{q}}{2}\cdot\nabla_{\pi_n}\right]\Psi^+(\pi_n,0)$$

$$= \prod_{n=1}^{A-1}\left(\frac{E(p_n)}{E(p_n^{[i]})}\right)^{\frac{1}{2}}\left[1 - i\chi_B\right]\Psi^+(\pi_n,0) \qquad (2.51)$$

where $\chi_B$ is

$$\chi_B = \frac{1}{2}\sum_{n=1}^{A-1}\left(\frac{-\mathbf{q}\cdot\pi_n}{4M} + \frac{\pi_n^2}{2m} + \frac{m\varepsilon}{M}\right)\frac{\mathbf{q}\cdot\rho_n}{2M} \quad + \quad \mathrm{hc}$$

$$= \chi\left(\theta = -\frac{\mathbf{q}}{2M}\right) + \frac{1}{2}\left(\frac{-\mathbf{q}\cdot\pi}{4M} + \frac{\pi^2}{2m} + \frac{m\varepsilon}{M}\right)\frac{\mathbf{q}\cdot\rho}{2M} + \mathrm{hc} \qquad (2.52)$$

Note that the energy factors in (2.51) are just sufficient to cancel the same factors in (2.46), and that the resulting boot phase, $\chi_B$, is equal to the $\chi$ of Eq. (2.30), plus an extra term involving $\pi$ and $\rho$ for the struck particle, where $\pi = -\Sigma\pi_\alpha$ and, for equal masses, $\rho = -\Sigma\rho_\alpha$. Finally, choosing $N^2$ to cancel the factors of $2\pi$, gives the following

$$F_A(q^2) = A\int \prod d^3\mathbf{p}_n \Big\{ \bar{\Psi}^{+}_{\lambda_n''}\left(\pi^{[f]}_n, 0\right)\left[1 + i\chi^{[f]}_B\left(\tfrac{1}{2}\mathbf{q}\right)\right] \times$$

$$\left(\frac{M_R}{P^0}\right)\left\langle j^0_R\right\rangle \left[1 - i\chi^{[i]}_B\left(-\tfrac{1}{2}\mathbf{q}\right)\right] \Psi^{+}_{\lambda_n'}\left(\pi^{[i]}_n, 0\right) D^{(\frac{1}{2})}_{\lambda_n'' \lambda_n'}\left(B_f^{-1} B_i\right)\Big\}$$

(2.53)

This is the general formula for the charge form factor of a nucleus with $A$ nucleons, including relativistic corrections to order $m^{-2}$.

Now (2.53) will be evaluated for the deuteron, where there is only one spectator with integration variable $\mathbf{p}_1$. Three quanities must be calculated: (i) the matrix element of the new charge operator, Eq. (2.48), (ii) the boost factor, $\chi_B$ , and (iii) the Wigner rotation. The new charge operator can be obtained from the general formula, Eq. (2.4), by substituting

$$\begin{aligned} p &\Rightarrow \pi^{[i]} = -\pi^{[i]}_1 = -\mathbf{p}_1 - \tfrac{1}{4}\mathbf{q} \\ p' &\Rightarrow \pi^{[f]} = -\pi^{[f]}_1 = -\mathbf{p}_1 + \tfrac{1}{4}\mathbf{q} \end{aligned} \qquad (2.54)$$

and adding the extra term given in (2.47). The result is

$$\left\langle j^0_R\right\rangle = G_E\left(1 - \frac{\mathbf{q}^2}{32m^2}\right) - \left(2G_M - \tfrac{3}{2}G_E\right)\left[\frac{i\sigma\cdot(\mathbf{q}\times\mathbf{p}_1)}{4m^2}\right] \qquad (2.55)$$

The factor multiplying $G_E$ is cancelled by the $(M_R/P^0)$ factor in (2.53), and the resulting one body current *does not agree with the results obtained in Eq. (2.7).* [To compare, rember that $\mathbf{p}_1 \Rightarrow -\mathbf{p}$ .] Before discussing the significance of this, the calculation of the other terms will be finished.

The Wigner rotation term gives another spin dependent correction factor [AC80]:

$$D^{(\frac{1}{2})}\left(R_2^{-1} R_1\right) = 1 - \frac{i\sigma\cdot(\mathbf{q}\times\mathbf{p}_1)}{8m^2} \qquad (2.56)$$

where the spin operator here is the spin of the spectator, and not the struck particle, as in

(2.55). However, the deuteron is symmetric in the spins, and hence (2.56) may be combined with (2.55) where it restores the factor of $2G_M - G_E$ obtained previously. The SO correction obtained in the two methods is seen to be identical!

Finally, the effect of the boost is easily calculated by transforming the integral to position space, using

$$
\begin{aligned}
&\left[1 - i\chi_B^{[i]}(-\tfrac{1}{2}\mathbf{q})\right]\Psi^{+}\left(\pi_1^{[i]},0\right) \\
&\qquad = \left[1 - i\chi_B^{[i]}(-\tfrac{1}{2}\mathbf{q})\right]\int \frac{d^3r}{\sqrt{(2\pi)^3}}\,\Psi^{+}(\mathbf{r})\, e^{+i(\mathbf{p}_1+\frac{1}{2}\mathbf{q})\cdot\mathbf{r}} \\
&\qquad = \int \frac{d^3r}{\sqrt{(2\pi)^3}}\, e^{+i(\mathbf{p}_1+\frac{1}{2}\mathbf{q})\cdot\mathbf{r}}\left[1 - i\chi_B'(-\tfrac{1}{2}\mathbf{q})\right]\Psi^{+}(\mathbf{r})
\end{aligned}
\tag{2.57}
$$

where $\chi_B'$ is obtained from $\chi_B{}^{[i]}$ of Eq. (2.52) by the replacement $\rho_1 = -\mathbf{r}$, and $\pi_1 = i\nabla_{\mathbf{r}}$. Note that the same formula holds for the final wave function, so that, performing the integration over $\mathbf{p}_1$, the new expression for the spin independent part of the form factor is

$$
F_D(q^2) = G_E^S \int d\mathbf{r}\; \bar{\Psi}^{+}(\mathbf{r})\left[1 + i\chi_B'(\tfrac{1}{2}\mathbf{q})\right] e^{i\mathbf{q}\cdot\frac{1}{2}\mathbf{r}} \left[1 - i\chi_B'(-\tfrac{1}{2}\mathbf{q})\right]\Psi^{+}(\mathbf{r}) \tag{2.58}
$$

As previously noted, the first term in $\chi_B'$ is identical to that obtained using the generator method, and gives the result labeled "B" in Eq. (1.6). The new term can readily be evaluated from (2.58), and gives *the missing DF terms in Eq. (1.6), plus the corrections* $\Delta_C$ and $\Delta_Q$ reported in Eq. (1.10) and (1.11).

## 2.4 Interlude: Discussion and Comparison

So far, the two methods give identical spin orbit corrections [Eq. (1.6) with $\Delta^{SO}$'s $=0$], but these corrections arise in different ways. In the *G-method*, the SO terms come entirely form the nucleon current; in the *C-method* they come partly from the Wigner rotations of the spin of the spectator, which occurs when the wave functions are boosted. However, the spin-independent corrections, which contribute only to the body form factors, $D_C$ and $D_Q$, are different (so far).

When these results were first compared, it seemed remarkable that the two methods gave identical results for the SO terms and "most" of the spin-independent terms. In the original *C-method* calculation [Gr66] very complicated formulas were obtained, and they were simplified only later [Fr73] when the *G-method* results were in hand. The extent of the agreement was gratifying, but the similiarity also emphasized the differences! The differences, contained in the $\Delta$ correction terms, can be reexpressed in terms of matrix elements of the *NN* potential by using the Schrodinger equation. For this reason they were sometimes referred to as the "potential" corrections, and much effort was devoted to attempts to derive them in the context of the *G-method.* Note especially the work of Coester and Ostebee [CO75], who exploited the fact that solutions different from (2.22) can be found, and of Glockle and Muller [GM81], who used a field theory model to determine the boost operator, and obtained results similar to the *C-method* ones presented here. These attempts were finally successful, and eventually [Gr78, Fr80] it was shown that the two methods, under reasonable assumptions, give identical results for *all* corrections to the deuteron form factors.

The modern derivation used in sections 2.2 and 2.3 shows the differences and similiarities of the two methods. Briefly, both determine the boost operator; one using the Poincare algebra for the generators, and one using the general form for the dependence of the relativistic wave function on covariant variables. The difference in the distribution of correction terms between nucleon current and nuclear boost is due, in part, to the fact that in the above treatment these corrections are evaluated in different frames in the two methods, and the separated corrections are not frame independent. The major difference is the appearance of the "extra" term in Eq. (2.52). Note that this "extra" term contributes *both* the SO correction *and* the $\Delta$ corrections, so the $\Delta$ corrections cannot be discarded without also loosing the well known SO correction. This suggests that some corrections are missing from the *G-method.* In the next sections it will be seen that this is true, and that there also additional corrections to the *C-method.* When all of these are taken into account, agreement can be reached.

## 2.5 Retardation Corrections

To set the stage for the discussion of retardation and pair term effects, return to the RIA defined in Fig. 9 above. For simplicity, consider a two body system only, and suppose that the *NN* interaction is described by a one boson exchange (OBE) model. In this case the two body bound state satisfies the equation shown diagramatically in Fig. 10a (and discussed in Part III below). The equation may then be used to rewrite the RIA form factor as shown in Fig. 10b. Through this series of arguments, one may consider the two Feynman diagrams shown in Fig. 11 to be the general definition of the current "operator" (consistent with OBE). Each of these relativistic diagrams can be decomposed into 6 *time ordered* diagrams [Gr76], as shown in Fig. 12. The focus of this section will be on the famous "recoil" graph, diagram (12c) which describes the interaction of the photon with the nucleon while the exchanged pion is "in flight". Because this contribution is included in the Feynman diagrams used to calculate the RIA [Wo75, Gr76], *it must not be added to other results obtained from the C-method discussed above.* However, retardation effects

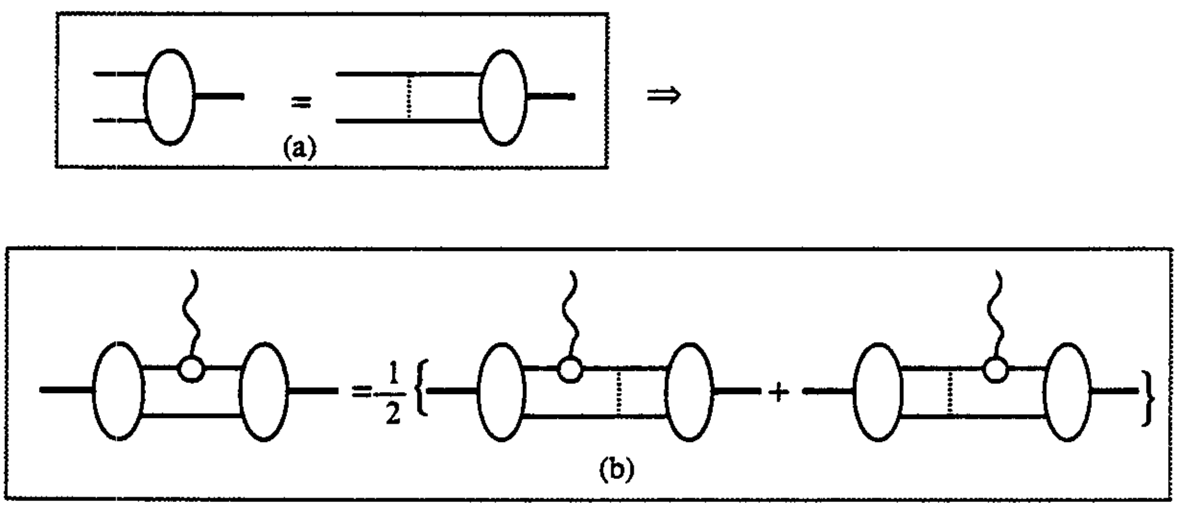


Fig. 10. (a) *Diagramatic representation of the OBE model for the NN interaction, and* (b) *use of this model to rewrite the RIA for the two body form factor.*

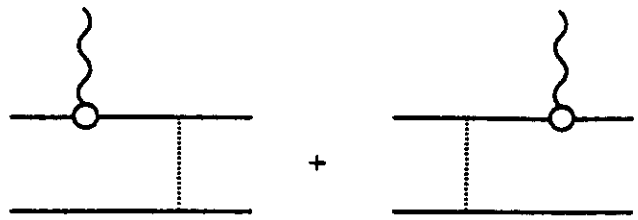

Fig. 11. *The relativistic current operator for the RIA, consistent with the OBE model, is given by the two Feynman diagrams shown in Fig. 10.*

have not been explicitly included in the treatment of the *G-method* developed so far, so in this case it is a new effect which must be added to the results previously obtained. It will give "potential" terms similar to those given in Eq. (1.11).

The first calculations of this effect [JL75, BR75] evaluated the diagram (12c). This gives a very large contribution, which is cancelled in lowest order completely, and largely cancelled even in higher orders [TH73, DW76]. This cancellation was systematically

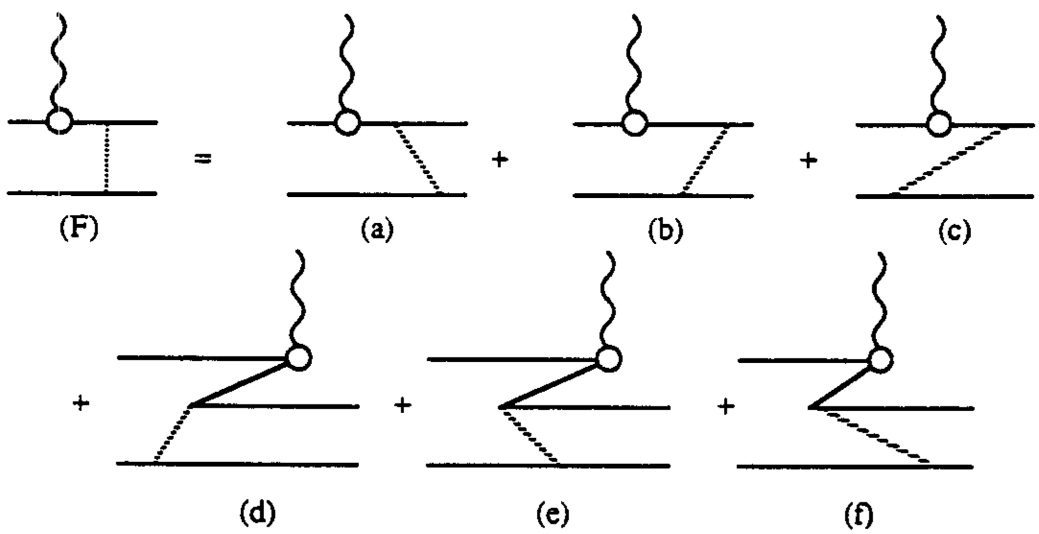


Fig. 12. *The single Feynman diagram shown above is equal to 6 time ordered diagrams. Diagrams* (a) *and* (b) *are time ordered diagrams describing the interaction of the initial state (time flows to the left),* (c) *is the "recoil" graph, and* (d) – (f) *are "pair" terms. The equality holds only if all external particles are on-shell; otherwise it is only approximate.*

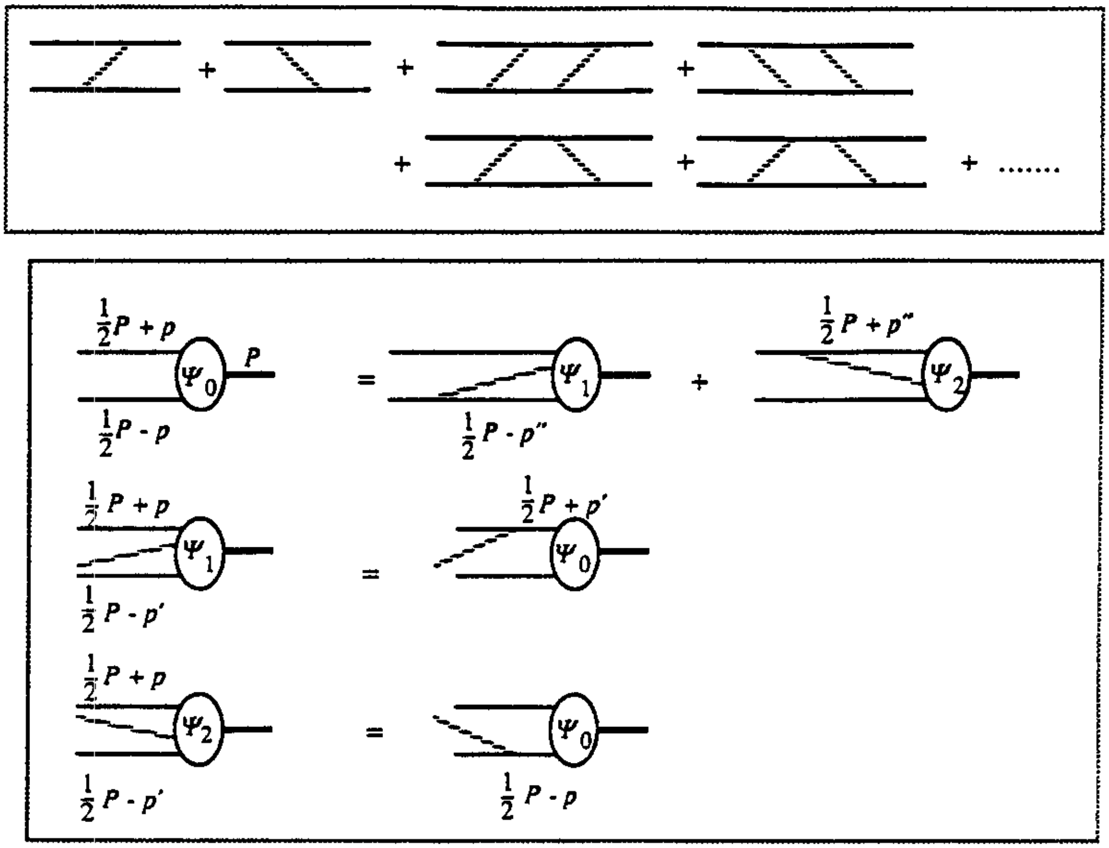


Fig. 13. *The infinite series of time ordered diagrams which are summed by the integral Eqs. (2.59) are shown up to $4^{th}$ order in the upper box. The lower box shows a diagramatic representation of the Eqs. (2.59), with momenta labeled.*

studied by Friar [Fr75b and 75c] and by Gari and Hyuga [GH76b, 77, and HG76].

The pion may be emitted by the first particle and absorbed by the second, as shown in Figs. 12b and 12c, or emitted by the second and absorbed by the first, as shown in Fig.

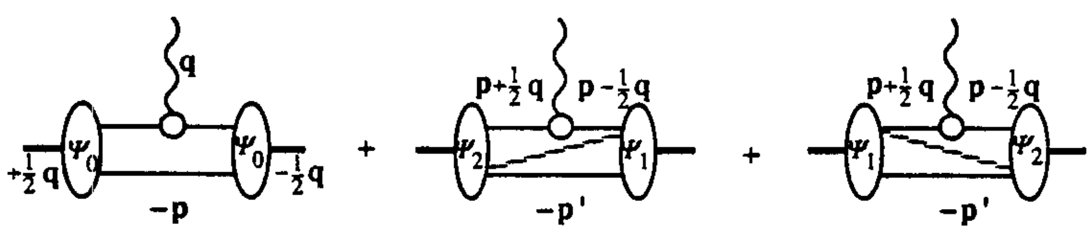


Fig. 14. Diagramatic representation of Eq. (2.60), with momenta labeled.

12a. The *NN* interaction generated from such time ordered diagrams is shown in Fig. 13. These diagrams are summed by the following set of coupled equations

$$(E_1 + E_2 - P_0)\,\Psi_0(\mathbf{p}) = \int \frac{d\,\mathbf{p}''}{(\sqrt{2\pi})^3\sqrt{2\omega}} \left\{ V_2(p''-p)\Psi_1(\mathbf{p},\mathbf{p}'') + V_1(p-p'')\Psi_2(\mathbf{p}'',\mathbf{p}) \right\}$$

$$(E_1 + E_2' + \omega - P_0)\Psi_1(\mathbf{p},\mathbf{p}') = \frac{V_1(p-p')}{(\sqrt{2\pi})^3\sqrt{2\omega}}\,\Psi_0(\mathbf{p}')$$

$$(E_1 + E_2' + \omega - P_0)\Psi_2(\mathbf{p},\mathbf{p}') = \frac{V_2(p-p')}{(\sqrt{2\pi})^3\sqrt{2\omega}}\,\Psi_0(\mathbf{p})$$

(2.59)

where $\Psi_0$ is the wave function for the *NN* channel, and $\Psi_1$ , $\Psi_2$ are wave functions for the $NN\pi$ channels where the subscript refers to the nucleon from which the last pion was emitted. In this language, the charge form factor arising from the NRIA plus recoil terms can be written

$$\begin{aligned} F_D = {} & 2\int d\mathbf{p}\;\Psi_0^\dagger(\mathbf{p} + \tfrac{1}{4}\mathbf{q}) \left\langle {}^{\rho}_{j} \right\rangle \Psi_0(\mathbf{p} - \tfrac{1}{4}\mathbf{q}) \\ & + 2\int d\mathbf{p}\; d\mathbf{p}' \left\{ \Psi_1^\dagger(\mathbf{p} + \tfrac{1}{4}\mathbf{q}, -\mathbf{p}' - \tfrac{1}{4}\mathbf{q}) \left\langle {}^{\rho}_{j} \right\rangle \Psi_2(\mathbf{p} - \tfrac{1}{4}\mathbf{q}, -\mathbf{p}' + \tfrac{1}{4}\mathbf{q}) \right. \\ & \left. + \Psi_2^\dagger(\mathbf{p} + \tfrac{1}{4}\mathbf{q}, -\mathbf{p}' - \tfrac{1}{4}\mathbf{q}) \left\langle {}^{\rho}_{j} \right\rangle \Psi_1(\mathbf{p} - \tfrac{1}{4}\mathbf{q}, -\mathbf{p}' + \tfrac{1}{4}\mathbf{q}) \right\} \end{aligned}$$

(2.60)

This equation is shown diagramatically in Fig. 14, where the momenta are also defined. The coupled equations (2.59) and the normalization condition [obtained from (2.60) at q = 0] can be cast into a convenient matrix form

$$P_0\Psi = H\,\Psi \qquad\qquad \Psi^\dagger \rho\,\Psi = 1 \tag{2.61}$$

where matrix multiplication includes integration over p, or {p , p' } if required, and the specific form of the matrices, in $1 \times 2$ block form, are

$$\Psi = \begin{pmatrix} \Psi_0(\mathbf{p}) \\ \Psi_\pi(\mathbf{p}, \mathbf{p}') \end{pmatrix} \qquad \rho = \begin{pmatrix} 1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

$$H = \begin{pmatrix} \left(E_1 + E_2\right) \delta(\mathbf{p} - \mathbf{k}) & V^\dagger(\mathbf{p}, \mathbf{k}'\mathbf{k})\, \sigma_1 \\ V(\mathbf{p}\,\mathbf{p}', \mathbf{k}) & \left(E_1 + E_2' + \omega\right) \delta(\mathbf{p} - \mathbf{k})\, \delta(\mathbf{p}' - \mathbf{k}') \end{pmatrix}$$

(2.62)

where the $2 \times 2$ sector is

$$\Psi_\pi(\mathbf{p}, \mathbf{p}') = \begin{pmatrix} \Psi_1(\mathbf{p}, \mathbf{p}') \\ \Psi_2(\mathbf{p}, \mathbf{p}') \end{pmatrix} \qquad V(\mathbf{p}\,\mathbf{p}', \mathbf{k}) = -\left(\frac{1}{(\sqrt{2\pi})^3 \sqrt{2\omega}}\right) \begin{pmatrix} V_1(p - p')\, \delta(\mathbf{p}' - \mathbf{k}) \\ V_2(p - p') \delta(\mathbf{p} - \mathbf{k}) \end{pmatrix}$$

(2.63)

Note that $H$ is not hermitian, because of the metric $\rho$, but that it does have the correct structure to preserve the norm (2.61). Introducing the new states, $\Phi$,

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \Psi \qquad a^\dagger a = \sigma_1 \tag{2.64}$$

and suppressing all reference to the momentum variables, the equations may be cast into a compact unitary form

$$P_0 \Phi = \mathbf{H} \Phi$$

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_0 & \mathbf{V}^\dagger \\ \mathbf{V} & \mathbf{H}_\pi \end{pmatrix}, \qquad \mathbf{V} = aV \tag{2.65}$$

To treat the recoil corrections properly, the orthogonality and normalization of the states must be preserved at the same time effects from the recoil graphs are calculated, and this requires diagonalizing (2.65). This procedure will cancel most of the recoil contributions, giving only a relativistic correction of order $m^{-2}$.

The solution to this problem is discussed in detail in GH76b. The unitary matrix which diagonalizes the problem is written in the form

$$\mathbf{U} = \begin{pmatrix} \left(1+\mathbf{F}^{\dagger}\mathbf{F}\right)^{-\frac{1}{2}} & -\mathbf{F}^{\dagger}\left(1+\mathbf{F}\,\mathbf{F}^{\dagger}\right)^{-\frac{1}{2}} \\ \mathbf{F}\left(1+\mathbf{F}^{\dagger}\mathbf{F}\right)^{-\frac{1}{2}} & \left(1+\mathbf{F}\,\mathbf{F}^{\dagger}\right)^{-\frac{1}{2}} \end{pmatrix} \tag{2.66}$$

where, to first order in the interaction, the operator **F** must satisfy the relation

$$\mathbf{H}_{\pi}\,\mathbf{F} - \mathbf{F}\,\mathbf{H}_0 = \mathbf{V} \tag{2.67}$$

Letting the **H**'s operate on their respective spaces, and denoting the resulting energies by $E_0$ and $E_{\pi}$, gives

$$\mathbf{F} = \mathbf{V}\left(\frac{1}{E_{\pi} - E_0}\right) \tag{2.68}$$

and the states, expressed in terms of an approximately diagonal *NN* state, $\phi_{NR}$, are

$$\Phi = \mathbf{U}^{\dagger}\begin{pmatrix} \phi_{NR} \\ 0 \end{pmatrix} = \begin{pmatrix} \left[1 - \frac{1}{2}\left(\frac{1}{E_{\pi}-E_0}\right)\mathbf{V}^{\dagger}\mathbf{V}\left(\frac{1}{E_{\pi}-E_0}\right)\right]\phi_{NR} \\ -\mathbf{V}\left(\frac{1}{E_{\pi}-E_0}\right)\phi_{NR} \end{pmatrix} \tag{2.69}$$

Restoring the original metric by undoing the transformation (2.64), gives finally

$$\begin{aligned} \Psi_0 &= \left[1-\tfrac{1}{2}\left(\tfrac{1}{E_{\pi}-E_0}\right)V^{\dagger}\sigma_1 V\left(\tfrac{1}{E_{\pi}-E_0}\right)\right]\phi_{NR} \\ \Psi_{\pi} &= -V\left(\tfrac{1}{E_{\pi}-E_0}\right)\phi_{NR} \end{aligned} \tag{2.70}$$

Substituting these into the form factor (2.60) gives

$$F_D = 2\int d\mathbf{p}\ \phi^{\dagger}_{NR}(\mathbf{p}+\tfrac{1}{4}\mathbf{q})\left\langle {\rho \atop j} \right\rangle \phi_{NR}(\mathbf{p}-\tfrac{1}{4}\mathbf{q}) + 2\int \frac{d\mathbf{p}\ d\mathbf{p}'}{(2\pi)^3}\ \phi^{\dagger}_{NR}(\mathbf{p}+\tfrac{1}{4}\mathbf{q})\ \ \rho_{\mathrm{recoil}}\left\langle {\rho \atop j} \right\rangle\ \phi_{NR}(\mathbf{p}'-\tfrac{1}{4}\mathbf{q}) \tag{2.71}$$

where the second term is the recoil correction, including the effects of making the two channels orthonormal. The correction factor in (2.71) is

$$\rho_{\mathrm{recoil}} = \frac{V_1 V_2}{(2\pi)^3\, 2\omega}\left(\frac{1}{\mathrm{D}}\right) \tag{2.72}$$

where the denominators are

$$\begin{aligned}\left(\frac{1}{\mathrm{D}}\right) = \Bigg[ & \underset{(a)}{\frac{1}{(E_3+\omega-E_1')(E_2'+\omega-E_2)}} + \underset{(b)}{\frac{1}{(E_3+\omega-E_1)(E_2+\omega-E_2')}} \\ & -\frac{1}{2}\Bigg( \underset{(c)}{\frac{1}{(E_3+\omega-E_1')(E_2'+\omega-E_2)}} + \underset{(d)}{\frac{1}{(E_3+\omega-E_1)(E_2+\omega-E_2')}} \\ & + \underset{(e)}{\frac{1}{(E_1+\omega-E_3')(E_2'+\omega-E_2)}} + \underset{(f)}{\frac{1}{(E_3+\omega-E_1)(E_1'+\omega-E_3')}}\Bigg)\Bigg]\end{aligned} \tag{2.73}$$

Evaluating $V_1$ and $V_2$, and expanding out the denominators in (2.73) gives finally

$$\rho_{\mathrm{recoil}} = -\left(\frac{g_\pi}{2m}\right)^2 \frac{\tau_1\cdot\tau_2}{2m\omega^4}\ \sigma_1\cdot(\mathbf{p}-\mathbf{p}')\ \sigma_2\cdot(\mathbf{p}-\mathbf{p}')\ \mathbf{q}\cdot(\mathbf{p}-\mathbf{p}') \tag{2.74}$$

[When the charge operator is added, this formula agrees with that quoted in GH76b.]

The next task is to evaluate this term for the deuteron. This is most easily done by Fourier transforming (2.71) to position space, and noting that the $\omega^{-4}$ term can be expressed as the derivative of the potential with respect to the pion mass. For the case of one pion exchange (OPE) without form factors, a short calculation gives the $J_0$ and $J_2$ contributions to the form factors reported in Eqs. (1.7) and (1.8). These are new corrections to be added to those previously obtained using the *G-method, but are not to be added to the results obtained using the C-method,* as already discussed above.

The occurance of the *NN* potential in these corrections suggests that these may be related to the terms (1.10) and (1.11), previously derived using the *C-method*. This is true. To compare, use the Schrodinger equation (in OPE approximation) to reduce the $I_0$ and $I_2$ terms in (1.11)

$$\hat{u}(r) = k_\pi\left[\mathrm{Y}_0(x)\,u(r) + \sqrt{8}\;\mathrm{Y}_2(x)\,w(r)\right]$$
$$\hat{w}(r) = k_\pi\left[\mathrm{Y}_0(x)\,w(r) + \sqrt{8}\;\mathrm{Y}_2(x)\left(u(r) - \tfrac{1}{\sqrt{8}}w(r)\right)\right] \quad . \tag{2.75}$$

It then turns out that these are identical to the $J_0 + J_0'$ and $J_2 + J_2'$ terms in (1.8)

$$I_0\Big|_{\mathrm{OBE}} = J_0 + J_0' \;, \qquad I_2\Big|_{\mathrm{OBE}} = J_2 + J_2' \tag{2.76}$$

and thus the $I_0$ and $I_2$ terms in may be interpreted in the language of the *G-method* as due, at least in part, to retardation effects. However, the *G-method* has produced only some of the additional terms; to obtain final agreement between (1.8) and (1.11) it is necessary to consider the "pair" corrections mentioned above. These contributions will be discussed in the next section.

## 2.6 Pair Contributions

Pair contributions arise from the electromagnetic interaction of the off-shell nucleon when it is in a negative energy state. The Feynman propagator insures that such a state propagates backward in time, and can be reinterpreted as an antinucleon, produced in the interaction, propagating forward. Figs. 12d-f are time ordered diagrams describing this process; the interaction produces a virtual nucleon-antinucleon pair for a brief instant. It is a remarkable fact that such contributions give significant corrections to order $m^{-2}$.

Both methods discussed above generate pair contributions, which will be discussed now. A new ingredient in both calculations is the matrix element on the nucleon current (2.1) between $u$ and $v$ spinor states. To lowest order, these matrix elements are

$$\begin{aligned}\left\langle j^0\right\rangle_{-+} &= \left(\frac{m^2}{E(p')E(p)}\right)^{\frac{1}{2}} \bar{v}(-p')\left[F_1(q^2)\gamma^0 + iF_2(q^2)\frac{\sigma^{0\nu}q_\nu}{2m}\right]u(p) \\ &\approx -\left[F_1\frac{\sigma\cdot(\mathbf{p}'-\mathbf{p})}{2m} + F_2\frac{\sigma\cdot\mathbf{q}}{2m}\right] \\ &\approx -\left[G_M\frac{\sigma\cdot\mathbf{q}}{2m} - G_E\frac{\sigma\cdot(\mathbf{q}-\mathbf{p}'+\mathbf{p})}{2m}\right] \approx -\left\langle j^0\right\rangle_{+-}\end{aligned} \tag{2.77}$$

where (2.77) holds for arbitrary $\mathbf{p}'$, $\mathbf{p}$, and $\mathbf{q}$. For use with corrections calculated with the *G-method*, where the spinors are evaluated in the Breit frame, $\mathbf{p}'-\mathbf{p} = \mathbf{q}$, so that the $G_E$ term cancels. However, when using the *C-method* where the wave functions were boosted to their rest frames [recall discussion leading to Eq. (2.55)], the $G_E$ term does not cancel. In this case, $\mathbf{p}'-\mathbf{p} = \mathbf{q}/2$ and the two terms combine just as in the spin orbit case. For the deuteron, the two cases give

$$\begin{aligned}\left\langle j^0\right\rangle_{-+} \approx -\left\langle j^0\right\rangle_{+-} &\approx_G -G_M\frac{\sigma\cdot\mathbf{q}}{2m} \\ &\approx_C -\left[2\,G_M - G_E\right]\frac{\sigma\cdot\mathbf{q}}{4m}\end{aligned} \tag{2.78}$$

In conclusion: with the *C-method* pair terms contribute only to the correction terms $\Delta^{SO}$ [Eq. (1.6)]; with the *G-method* they contribute to all of the $\Delta$'s.

The specific form of the pair term corrections can be easily extracted from Eq. (2.53). In terms of the wave function for the negative energy channel, defined in Eq. (2.45), the lowest order result for a general nucleus is

$$F_A(q^2)_{pair} = A \int \prod d^3p_n \Big[ \bar{\Psi}^-\left(\pi_n^{[f]},0\right) \left\langle j_R^0 \right\rangle_{-+} \Psi^+\left(\pi_n^{[i]},0\right)$$

$$+ \bar{\Psi}^+\left(\pi_n^{[f]},0\right) \left\langle j_R^0 \right\rangle_{+-} \Psi^-\left(\pi_n^{[i]},0\right) \Big] \tag{2.79}$$

Evaluation of this result requires a calculation or estimate of the negative energy wave functions. These components for the deuteron have been systematically studied [Re72, HG73, and BG79]. In analogy with the Dirac wave function for the hydrogen atom, which has a large $S$ state upper component and a small $P$ state lower component, the deuteron has two small $P$ state components (spin singlet, $s$, and triplet, $t$) which are companions to its large $S$ and $D$ state terms. These smaller components can be written in a convenient matrix form [BG79]:

$$\phi_i^-(\mathbf{r}) = \chi_2^\dagger \, \phi_i^-(\mathbf{r}) \; \chi_1$$

$$\phi_i^-(\mathbf{r}) = i\sqrt{\frac{3}{4\pi}} \left[ \frac{v_s(r)}{r^2} \, \mathbf{r}\cdot\xi_i - \frac{1}{\sqrt{2}} \frac{v_t(r)}{r^2} \left( \sigma\cdot \mathbf{r} \, \sigma\cdot \xi_i - \mathbf{r}\cdot\xi_i \right) \right] \frac{i\sigma_y}{\sqrt{2}} \tag{2.80}$$

Writing the matrix element (2.79) for the deuteron, and Fourier transforming to position space, gives the contributions to the $\Delta^{SO}$'s defined in Eqs. (1.10) and (1.11)

This completes the analysis of the $m^{-2}$ corrections to the deuteron electric form factors using the *G-method*. The total result, given in Eqs. (1.6), (1.10), and (1.11), has now been obtained.

To compare with the *G-method* results, the coupled equations which give the $v$'s in terms of $u$ and $w$ can be solved in the lowest order OPE approximation [HG73, Gr74, and Gr78]. The solution depends on the type of pion coupling used. If the relativistic coupling is a mixture of *ps* and *pv* terms

$$\Lambda_\pi = g_\pi \left[ \lambda \, \gamma^5 + \frac{(1-\lambda)}{2m} \gamma^5 \gamma^\mu (p_f - p_i)_\mu \right] \tag{2.81}$$

where the mixing parameter $\lambda$ is defined so that the coupling (2.81) is independent of it if *both* nucleons are on mass-shell, then the solution is

$$v_t = \sqrt{3}\ \lambda \left(\frac{k_\pi}{m_\pi\, m}\right) Y_1(x)\, [\sqrt{2}\, u + w]$$

$$v_s = \sqrt{3}\ \lambda \left(\frac{k_\pi}{m_\pi\, m}\right) Y_1(x)\, [u - \sqrt{2}\, w] \tag{2.82}$$

Using $\lambda = 1$, for pure *ps* coupling (to be discussed further below), inserting these into (1.11) and doing some rearranging, gives

$$I_0^{SO}\Big|_{\mathrm{OBE}} = J_o\,' , \qquad I_2^{SO}\Big|_{\mathrm{OBE}} = J_2\,' \tag{2.83}$$

Thus, separating the terms into $G_E$ and $G_M$ contributions, for the moment, *the $G_E$ contributions from the pair terms calculated in the C-method, (2.83), cancel some of the $G_E$ contributions previously calculated, (2.76), giving exactly the same result as the $G_E$ corrections obtained in the G-method approach.* Since the pair terms from the *G-method* have not yet been estimated, but since they depend only on $G_M$, the $G_E$ corrections from both methods have now been found to be the same (in OPE approximation with *ps* coupling).

Evaluation of the pair corrections in the *G-method* requires that the time ordered diagrams shown in Figs. 12d and 12f (and their counterparts with all lines reflected) be evaluated. The other diagram, 12e, is smaller because it involves two energy denominators with an *N-Nbar* pair, while the others have only one. Using the labeling for momenta given in Fig. 14, the operator is

$$\left\langle j^0 \right\rangle_{\mathrm{pair}} = \frac{\lambda\, g_\pi^2}{8\, m^3\, \omega^2}\ \sigma_2 \cdot (\mathbf{p} - \mathbf{p}')\ \sigma_1 \cdot \mathbf{q} \left[ G_M^S\ \tau_1 \cdot \tau_2 + G_M^V\ \tau_{2z} \right] + (1 \leftrightarrow 2) \tag{2.84}$$

[This operator agrees with [GH76a], which is consistent with [HG76], if their $c$=1, $d$=−1, but disagrees with the Hannover group [SH87], who use $c$=1, $d$=0.] The simplist way to evaluate this contribution to the deuteron form factors is to cast the matrix elements [of the form Eq. (2.60)] into position space, where the integrals can be simplified by integration by parts, and the $Y_1$ function appears naturally. The $J_0'$ and $J_2'$ terms reported in Eq. (1.8) are then obtained. Hence, with this final contribution, the results obtained from the *G-method* and *C-method* agree (in the OPE approximation with pure *ps* coupling).

## 2.7 Conclusions

After considerable effort, two different methods have been found to give the same lowest order relativistic corrections to the deuteron electric form factors [Fr77a, 80, and Gr78]. The agreement is satisfying and gives considerable insight into the nature and origin of relativistic effects. The final agreement was achieved only after all effects were taken into account, leading to the conclusion (iv) drawn in section 1.6 above. Furthermore, effects identified with a single physical origin do not appear in the same manner in the two methods, showing that attempts to classify and name relativistic effects can have only limited value.

However, further reflection shows that the agreement between the two methods is less significant than it appears at first sight. While all of the contributions obtained from the *C-method* were calculated used the same theoretical framework, this was not the case for the *G-method*. In this approach, the algebra satisfied by the Poincare group was used to calculate boost effects, and the time ordered formalism was used to calculate the recoil and pair term contributions. Unfortunately, some additional assumptions about how to join these two parts of the calculation are needed before a final answer can be obtained. For example, it turns out the the solution (2.27) for the interaction term in the boost operator is not unique; other choices are possible and this non-uniqueness has been discussed extensively in the literature. It must be *assumed* that the solution (2.27) is the correct one to use with the retardation effects calculated from the time ordered analysis discussed in section 2.5. If we had used a solution different from (2.27), we would have obtained a different answer. Coester and Ostebee [CO75] did just that, and were able to obtain retardation effects *consistent with the C-method results without inclusion of any recoil corrections*. Adding the recoil corrections calculated in section 2.5 to the CO75 result would have given an answer twice as large.

Friar [Fr77a, 80] has discussed this ambiguity in terms of the freedom to use a unitary transformation to redefine the wave functions. For example, if the wave function $\Psi$ is transformed by a unitary transformation $U$

$$\Phi = U\Psi \tag{2.85}$$

then matrix elements of a physical quanity $O$ are related by

$$\left\langle \Psi^{\dagger} O \; \Psi \right\rangle = \left\langle \Phi^{\dagger} \; O' \Phi \right\rangle \; , \qquad O' = U \; O \; U^{\dagger} \tag{2.86}$$

In this way relativistic effects may be shifted in or out of the wave function, and if there is no information about the dynamical content of the wave function, it is impossible to decide which operator to use, and it is possible to get any effect one wishes. Friar [Fr80] writes his operators in terms of two parameters which describe this freedom. This has lead to much uncertainty in the community over what corrections to apply. Recently, Truhlik and Adam [TA89], in their study of threshold electrodisintegration, observed that this uncertainty is large numerically [see their Fig. 8].

Fortunately, the *C-method* does not suffer from such ambiguities, at least as long as the relativistic wave functions defined by the method are used to estimate the relativistic corrections. Furthermore, the corrections obtained from the *C-method* show how the corrections obtained from the *G-method* should be joined, or, in Friar's language, they fix the representation. This is even the case for the differences between *ps* and *pv* coupling for the pion; the form of the result (1.11) holds for both couplings, but the numerical value of the integrals is quite sensitive to the nature of the coupling because the size of the negative energy wave functions are [recall Eq. (2.82)]. We conclude that the *C-method* is the only one of the two which can yield answers systematically and without ambiguity, and are thus lead to one reason why fully covariant methods are needed in future work.

## III. Covariant Methods

A number of covariant relativistic equations have been developed for the dynamical treatment of few body systems and their interactions. These include the Bethe-Salpeter equation [SB51], the Blankenbecler-Sugar equation [BS66, LT63], and the spectator (sometimes referred to as the Gross) equation [Gr69]. Applications of the Blankenbecler-Sugar equation to the treatment of form factors can be found in JW84. A few remarks about the light-front (sometimes referred to as the Weinberg) equation [We66, and see JK88], not discussed in this chapter, will be given below. Other equations are described in Refs. To71, WJ73, and WM89. This Part will review recent work using the Bethe-Salpeter (BS) and spectator (G) equations to describe the two and three nucleon systems and their interactions.

### 3.1 Introduction and Comparisons

The equations we will discuss are linear integral equations. For the scattering amplitude $M$ they have the following form

$$\begin{aligned} M(p\,p',\, P) &= V(p\,p',\, P) - \int dk\, V(p\,k,\, P)\; G(k,\, P)\; M(k\,p',\, P) \\ &= V(p\,p',\, P) - \int dk\, V(p\,k,\, P)\; G(k,\, P)\; V(k\,p',\, P) \\ &\quad + \int dk\, dk'\, V(p\,k,\, P)\; G(k,\, P)\; V(k\,k',\, P)\;\; G(k',\, P)\; V(k'\,p',\, P)\; \ldots \end{aligned} \tag{3.1}$$

where $V$ is the kernel, or relativistic potential, and $G$ is the propagator for the $NN$ or $NNN$ system, and for a two nucleon system,

$$\begin{aligned} P &= p_1 + p_2 & p_1 &= \tfrac{1}{2}P + p \\ p &= \tfrac{1}{2}(p_1 - p_2) & p_2 &= \tfrac{1}{2}P - p \end{aligned} \tag{3.2}$$

with $P$ the total, and $p$ the relative, four-momentim of the system. As the second line in

(3.1) suggests, a covariant equation can be regarded as a tool for summing an infinite class of Feynman diagrams, and must be used whenever the physics requires such a sum. This is necessary when treating bound states, because a bound state appears as an $s$ channel pole in the $M$ matrix, and if it is truly composite, no finite number of Feynman diagrams has such a pole. It is also necessary when treating elastic scattering at threshold where it is important to treat unitary exactly; again, no finite number of Feynman diagrams is unitary. Of course, when the kernel is strong, the solutions of the equation will exist when the corresponding series diverges, and the solution can then be regarded as the analytic continuation of the infinite series from a region where it is properly defined to one where it diverges. But the series is a useful starting point for several reasons: (i) it is closely connected to field theory and provides answers to dynamical questions which may arise, (ii) it tells how to construct, or derive, the equation, and (iii) it shows that the equation is covariant because the diagrams on which it is based are covariant.

Before continuing with this discussion, it is well to clarify what is meant by "covariant". If the interacting nucleons are off-shell, which is always the case for equations of the type described in (3.1), the decomposition of its kernel (usually a finite sum of relativistic Feynman diagrams) into a finite number of *time ordered* diagrams, which was illustrated in Fig. 12, does *not* hold. As a consequence, no equation with a kernel consisting of a finite number of time ordered diagrams is covariant. This also holds for the $\tau$ ordered diagrams encountered in the light front approach; strictly speaking, the light front equation is not covariant. In is not manifestly rotationally invariant, and dynamical constraints must be imposed in order to obtain rotational invariance [see, for example, JK89, and references therein].

The bound state equation can be derived from the scattering equation by postulating that a bound state will show up as a pole in scattering matrix

$$M(p\,p';\,P) = -\frac{\Gamma(p,P)\;\bar{\Gamma}(p';\,P)}{m_B^2 - P^2} + R(p\,p';\,P) \tag{3.3}$$

where $\Gamma$ is the bound state vertex function, and $R$ is a remainder function regular at the bound state pole. The relativistic bound state wave function is defined by

$$\Psi(p,P) = N^{-1}\, G^{-1}(p, P)\; \Gamma(p, P) \quad , \tag{3.4}$$

where $N$ is a normalization constant. Substituting (3.3) into (3.1) and requireing that the equation hold at the pole, gives the bound state equation

$$G^{-1}(p, P)\, \Psi(p,P) = -\int dk\, V(p\,k,\ P)\ \Psi(k,P) \tag{3.5}$$

and the normalization condition for the relativistic wave function

$$\begin{aligned} 1 = \int dk\, \bar{\Gamma}(k, P) \left.\frac{\partial G(k, P)}{\partial P^2}\right|_{P^2=m_s^2} \Gamma(k,P) \\ - \iint dk\, dk'\ \bar{\Psi}(k,P) \left.\frac{\partial V(k\,k';\ P)}{\partial P^2}\right|_{P^2=m_s^2} \Psi(k',P) \end{aligned} \tag{3.6}$$

The definition of the domain of integration $dk$ and the propagator $G$ differs for each equation. The BS equation is defined by

$$\int dk \Rightarrow \int \frac{d^4 k}{(2\pi)^4} \qquad G(k,P) = \frac{(m+\not{k}_1)_1\,(m+\not{k}_2)_2}{(m^2-k_1^2-i\varepsilon)(m^2-k_2^2-i\varepsilon)} \tag{3.7}$$

where the subscripts on the projection operators are a shorthand notation for the Dirac indicies of each particle. The G equation, for the case with particle 1 on-shell, is defined by

$$\int dk \Rightarrow \int \frac{d\,\mathbf{k}_1}{(2\pi)^3}\left(\frac{m}{E(\mathbf{k}_1)}\right) \qquad G(k,P) = \left(\frac{m+\not{k}_1}{2m}\right)_1 \left.\frac{(m+\not{k}_2)_2}{(m^2-k_2^2-i\varepsilon)}\right|_{k_1^2=m^2} \tag{3.8}$$

with $k_1{}^2 = m^2$ treated as a constraint in all kernels and wave functions, and the kernel must be explicitly antisymmetrized in order to insure that the Pauli principle is satisfied. The covariance of the G equation is easily displayed using the identity (2.40). The constraint means that the projection operator for particle 1 can be replaced by the sum over $u$ spinor states only, so that the states have only $4 \times 2 = 8$ spin components instead of the $4 \times 4 = 16$ components of the BS equation. In addition, the internal energy is fixed by the mass-shell constraint, which is, in the CM

$$k^0 = E(\mathbf{k}) - \tfrac{1}{2}W \tag{3.9}$$

with the convention that $W = P^0$ in the CM. This means that solutions of the G equation have the same number of momentum variables as in the nonrelativistic case, differing only in the presence of the extra spin components associated with the negative energy degrees of freedom of the off-shell particle. The G equation can therefore be written in a form similar to a Schrodinger equation, except with relativistic kinematics and additional channels resulting from the the negative energy degrees of freedom.

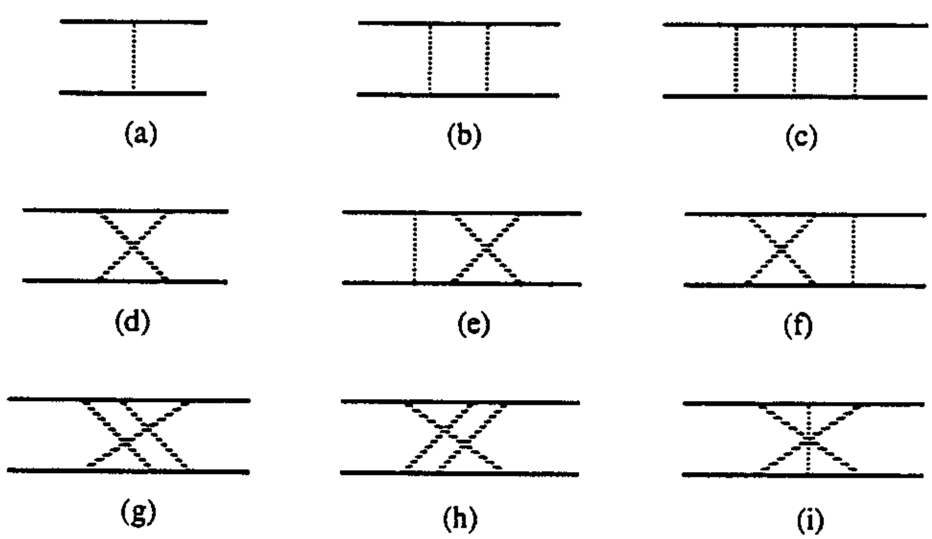


Fig. 15. *Ladder and crossed ladder diagrams to 6^th order in the meson-nucleon coupling constant. Diagrams (a)-(c) are ladder diagrams; the rest are crossed ladders. With respect to the BS equation, only (a), (d), and (g)-(i) are irreducable.*

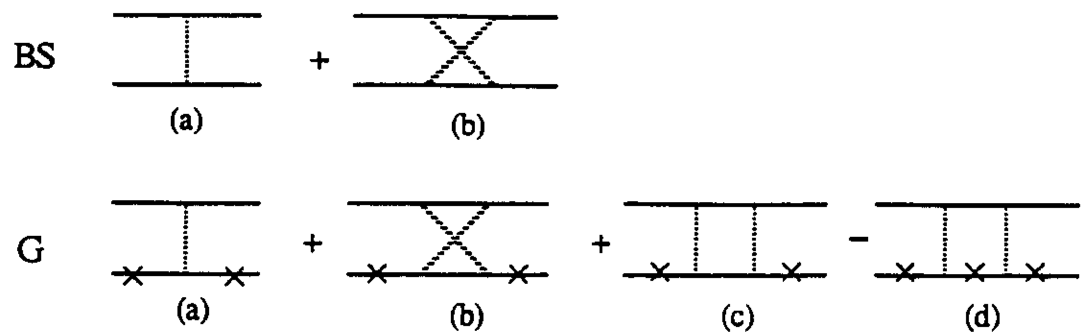


Fig. 16. *Irreducable kernels to $4^{th}$ order for the BS and G equations. The $\times$ on the lower line means the particle is on-shell. For the G equation, diagrams (c) and (d) together can be regarded as a single diagram which adds the difference between the full box diagram (c) and the iteration of the $2^{nd}$ order kernel (d).*

Once the equation is chosen, its kernel is fixed by the choice of the infinite class of Feynman diagrams one wishes to sum. This choice is determined, in turn, by the physics. For the treatment of elastic scattering below the meson production threshold, a minimal choice of diagrams is the sum of all ladder and crossed ladder diagrams [WJ72], shown in Fig. 15. In order to sum these completely, the kernel must be an infinite sum of all those diagrams which are *irreducable*, which means they cannot be constructed by *iterating lower order terms* in the kernel. It is easy to see how this works for both the BS and G equations, and the $4^{th}$ order irreducable kernels corresponding to each equation are shown in Fig. 16. Note that the $4^{th}$ order kernel for the G equation requires two terms (diagrams (c) and (d) will be lumped together as a single term), while only one is required for the BS equation.

The G equation is only one example of an infinite class of relativistic equations which all have the property that they reduce the four-dimensional integration of the BS equation to three dimensions [WJ73, Gr82a]. These equations are sometimes referred to as quasipotential (QP) equations.

The advantages and disadvantages of the BS equation, or any of the QP equations, has been the focus of some study [WJ73, MG78, ZT81, Gr82a], and remains a topic of considerable interest. A definitive answer can be given in the special case when one particle has spin zero, isospin zero, and is heavier than the other. Denoting the mass of the spin zero particle by $M$, and the lighter particle (any spin) by $m$, we expect that, as $M \rightarrow \infty$, the *two body equation should reduce to the exact one body equation* for the lighter particle moving in the instantaneous potential created by the heavier. This requirement, which will be called the "one-body limit", emerges from the physical idea that a particle with no internal structure (zero spin and isospin), should decouple from the description of the lighter particle as its mass becomes infinite. It can be easily seen that the G equation has the one body limit. Its kernel, in OBE approximation, already gives the one-body limit:

$$V(pk,\ P) = \frac{g_1 g_2 O_2}{\omega^2(\mathbf{p}-\mathbf{k}) - [\,E_1(p) - E_1(k)]^2} \quad \Rightarrow \quad \frac{g_1 g_2 O_2}{\omega^2(\mathbf{p}-\mathbf{k})} \quad . \tag{3.10}$$

To see why this happens, look at the irreducable kernels for the G equation shown in Fig. 16. Examination of the three 4$^{th}$ order terms [diagrams (b)-(d)] shows that they *cancel* each other as $M \rightarrow \infty$, and furthermore, it can be shown [Gr82a] that this *cancellation occurrs to all orders*. Hence, in the $M \rightarrow \infty$ limit, the simple kernel (3.10) sums all ladders and crossed ladders *exactly*. The BS equation does not have this property. *Each* term in the infinite series for the BS kernel remains *finite* as $M \rightarrow \infty$, and therefore *all* (an infinite number) are needed to recover the one body limit. From this point of view, the BS equation is ineffecient in summing ladders and crossed ladders, and does not have a smooth nonrelativistic limit.

However, the cancellations leading to the one body limit have not been proved for the general case of charged meson exchange between two spin $^1/_2$, isospin $^1/_2$ particles of *equal* mass (the *NN* system), and the massive spin zero example described above is far removed from this practical case of interest! Until more is known, interest in the use of both equations to describe the *NN* system is high.

## 3.2 Current Conservation

Before any equations can be applied to the study of electromagnetic interactions of few body systems, it is necessary to know how to construct a gauge invariant interaction in order to insure that current is conserved. This is also a requirement of nonrelativistic theories, of course, but the demand is somewhat more critical when the theoretical approach purports to be more fundamental. The problem is almost trivially solved for isospin zero elastic form factors, where the symmetry of the system allows a proof that even the RIA is gauge invariant [ZT80]. The full problem cannot be avoided when dealing with inelastic processes, and for many years it was a serious obstacle to progress. The issue was addressed by de Forest [dF83], who developed a prescription for modifying the one body current so that the impulse approximation conserves current, but the solution introduces some arbitrary features and is not entirely satisfactory.

While the general problem is not solved, and there is still not a fully satisfactory way of conserving current for the $(e, e'p)$ reaction from complex nuclei (for example), a method was recently found [GR87] for treating the few body problem which works *even in the presence of phenomenological strong form factors*. The method also permits the use of different (appropriate) electromagnetic form factors for each of the particles with which the photon interacts, so that, for example, the experimentally determined nucleon form factors may be used in the one nucleon current, the pion form factor may be used for the "true" pion exchange interaction, and $F_1$, or the axial form factor, $F_A$, may be used for the pair contributions. The common belief that gauge invariance requires a common electromagnetic form factor, or that a gauge invariant calculation is impossible without a microscopic theory of the structure of the meson-nucleon vertices, is false.

Briefly, the construction of the conserved current requires that (*a*) the *single* nucleon or meson currents satisfy an appropriate Ward-Takahashi identity, (*b*) the interaction part of the current operator be constructed from the irreducable kernel by coupling the photon to all possible places in the kernel, and adding appropriate contact terms wherever there are momentum dependent couplings, and (*c*) the initial and final relativistic wave functions in the matrix element be calculated using the same relativisitic equation with the *same* kernel. The last requirement makes this method inappropriate for complex systems, and the second places a premium on using simple kernels (OBE, for example). Finally, the method requires that the strong form factors be written as a product of functions, each of which can depend on the (virtual) mass of only *one* of the particles entering and leaving the vertex. Furthermore, if a particle is involved in more that one vertex (as the pion is in the $\pi NN$ and $\pi\Delta N$ couplings, for example) the function carrying the pion mass dependence must be the same in both cases (a kind of universality requirement). These conditions are satisfied in the typical case of form factors which depend on the meson mass only.

The remaining sections describe some recent results using the BS and G equations.

## 3.3 Applications of the Bethe-Salpeter Equation

The BS equation has been applied by Tjon and his collaborators to the *NN* and $\pi NN$ coupled systems, to the three nucleon problem, and to the description of two and three body form factors. Some of this work has been reviewed at international conferences [Tj85, 87], and is published in various summer schools [Tj86a, 86b, 89, and90]. Related work on the scattering of polarized nucleons from nuclei is not reveiwed in this chapter.

The *NN* problem was first studied [FT75] using a OBE model consisting of six mesons: $\pi$, $\rho$, $\omega$, $\varepsilon$ (an isoscalar, scalar meson), $\delta$ (an isovector, scalar meson), and $\eta$. Form factors were used at the meson-nucleon vertices. Using the helicity representation, the equation can be expanded in partial waves, much as is done for the nonrelativistic problem, with the Wigner spin rotation matrices replacing the usual spherical harmonics [Ku72]. Because of the extra degrees of freedom, there are in general 8 coupled partial wave states instead of the 2 which occurr in the nonrelativistic coupled spin triplet channels. The resulting "radial" equations are two dimensional, and can be cast into a Euclidean form by a Wick rotation [Wi54]. By adjusting the meson parameters, and using *pv* coupling for the pion, a reasonable fit to the *NN* phase shifts up to 250 MeV can be obtained. One interesting feature of the results is that the negative energy states are replusive in all partial wave channels $J < 2$, except for the $^1S_0$ channel, in which they supply an added attraction. This result is qualitatively different from that found using the G equation, where the negative energy channels are always repulsive. A realistic deuteron wave function was obtained, and was used to calculate the deuteron form factors [ZT80, 81], giving results reviewed earlier. A more recent version of the same model with meson exchange contributions [HJ89] does quite well, and results were described in section 1.1 [see Fig. 3].

To extend this work to intermediate energies, above the pion production threshold, $\pi NN$ intermediate states were included by adding $N\Delta$ channels, first with the $\Delta$ given a width dependent only on the total $NN$ energy [vT84], and later with the $\Delta$ propagator modified so as to satisfy three body unitarity [vT86]. Fits to the $NN$ scattering inelasticities come out somewhat too low, but are improved by adding the $\pi d$ channel, and the $P$ wave phase shifts are not well fit in the neighborhood of 800 MeV. To obtain the best fits, it is necessary to use a form factor at the $N\Delta\pi$ vertex with a strong cutoff, corresponding to a large spatial size, and the form factor must depend on the *relative* momentum between the pion and nucleon. This latter feature means that the recent method, reviewed in sec. 3.2, for constructing electromagnetic interactions which are gauge invariant in the presence of strong phenomenological form factors cannot be used in this case.

Recently, progress has been made with the relativistic three nucleon problem. The BS equation for three scalar particles has been solved using a rank-one separable potential, an approach which neglects all $D$ state admixtures and treats spin in a nonrelativistic way, and the solution applied to the calculation of three nucleon form factors [RY88, Ru90]. At the same time, rank-one separable solutions were found for the nonrelativistic problem, and two different quasipotential equations, and the results compared to those obtained for the BS equation. The binding energy obtained for the triton was greater using the BS equation than for the other equations, with the corresponding form factor more rapidly decreasing with $Q^2$, as expected. A multirank separable approach, neglecting negative energy states but retaining spin degrees of freedom for the positive energy sector, has also been used for the deuteron and the deuteron form factors [RT90]. Results similar to other work on the deuteron have been obtained.

Work using the BS equation to describe the electromagnetic interactions of few body systems is proceeding at a rapid rate. Numerical solutions of the equation do not pose the problems they once did, and a number of calculations, which can be compared to data and other relativistic approaches, will be available in the next few years.

## 3.4 Applications of the Spectator Equation

As described in section 3.1 above, the spectator (G) equation has a propagator which restricts one nucleon to its mass-shell [Gr69]. It is designed to have a smooth nonrelativistic limit, and these features have already been reported in sections 1.5 and 3.1 above [Gr74]. A OBE model for the deuteron was solved exactly [BG79], and the solutions used to calculate the deuteron form factor [AC80], but the OBE model used to determine these deuteron wave functions was not fit, at the same time, to the rest of the *NN* scattering data, so the wave functions obtained were, to some extent, unrealistic. Recently an accurate fit to the *NN* data below 300 MeV was obtained [GV90], and the resulting wave functions are being used to calculate electromagnetic observables.

A novel feature of the OBE model used in all of these calculations is the $\pi NN$ coupling, which has the mixed form given in Eq. (2.81). The mixing parameter, $\lambda$, gives the fraction of the total coupling which has the *ps* structure, with $1-\lambda$ giving the remaining fraction which is *pv*, and the two terms normalized so that the total coupling is independent of $\lambda$ if *both* the initial and final nucleons are on-shell. When one nucleon is in a negative energy state, however, the coupling is very sensitive to $\lambda$, being directly proportional to it in the nonrelativistic limit. One of the goals of the recent work [GV90] was to systematically study this sensitivity to $\lambda$. To this end, two, equally good fits to the *NN* data were obtained. For one case (referred to as Model II) $\lambda$ was fixed at 0 (corresponding to pure *pv* pion coupling) and the parameters of the 6 mesons previously used in the BS model were varied. For the second case (referred to as Model I), $\lambda$ was allowed to vary in the fitting process, but only four mesons were used. The goal was to see if the results suggested by the nonrelativistic "fits" to the Reid potential [Gr74] could be reproduced in a careful fit to actual data, and confirm that the additional freedom introduced by $\lambda$ would make it possible to fit the data using only the four mesons believed to be really essential to any OBE description of *NN* scattering (specifically, the $\pi$, $\sigma$, $\rho$, and $\omega$). The original nonrelativistic "fits" gave $\lambda = 0.4$, while the result obtained for Model I gave $\lambda = 0.23$. It appears that an admixture of *ps* pion coupling is helpful to the phenomenology, and that the low energy data can be accurately fit by only four mesons.

Putting one particle on the mass-shell gives the G equation a very unsymmetrical structure, and the Pauli principle cannot be recovered by antisymmetrizing the final solutions unless the kernels are antisymmetrized from the beginning. If this is done, however, the Pauli principle is exactly satisfied, and all recent solutions have been obtained with antisymmetrized kernels. A second technical problem is that the procedure of putting one particle on mass-shell tends to introduce unphysical singularities into the kernels of the equation [ZT81, Gr82a]. These have their counterpart in the BS equation, and are avoided there by performing the Wick rotation, but a similar method has not yet been found for the G equation. These singularities disappear if *either* the initial *or* final state is on-shell (i.e. both particles on-shell), so no physical matrix element is singular because of them. Furthermore, they cancel if the kernel is treated to all orders, so droping their imaginary parts is justified, giving real, hermitian kernels. The remaining principal value singularities are small and distant from physical regions, and recent studies [GV90] show that they give negligible numerical contributions. They remain an inelegant feature of the G equation, but seem to be unimportant otherwise.

The spectator equations for the three body system have been written down [Gr82b], and decomposed into partial waves [Gr83]. The extension of the spectator idea to the three body sector requires that two of the three particles be restricted to their mass-shell, producing fully covariant equations of the Faaddev type with *the same number of continuous variables as in the nonrelativistic case*. The equations therefore have the same structure as the corresponding nonrelativistic equations, but with a doubling of channels, as in the two body case. The two body amplitude which drives the equations satisfies the *same* equation as it did in isolation, so the cluster decomposition property is exactly satisfied. A systematic proceedure for applying the spectator formalism to a variety of electromagnetic processes is being developed [for reviews, see Gr86, 88, 89a, 89b, and 90]. More results from this method will be available in the next few years.

## 3.5 Conclusions

Discussion and conclusions to various topics covered in this chapter have already been given in sections 1.6, 2.4, and 2.7. To conclude this Part and the full chapter, I emphasize that fully covariant methods for treating few body systems (using either the BS or G equations) are now well developed, and can be used to give a consistent, gauge invariant description of electromagnetic interactions of few body systems. Both equations are closely tied to field theory, and can therefore base their phenomenology on a well defined dynamics. Older, expansion methods which try to graft relativistic effects onto nonrelativistic theories have taught us much about the physics (and were reviewed so thoroughly in Part II for this reason), but cannot address the next generation of problems, which are concerned with achieving a high level of internal consistency. Nonrelativistic models, if they are phenomenological, are not able to tell us unambigously what additional effects must be added to them simply because they do not tell us enough about what effects they already contain. And, there is no longer any reason to use phenomenological models (at least for few-body systems) because covariant models can be fitted to data just as well, and give us much more information about how to extend them *consistently* to other processes. As experiments and theory go to the higher momentum transfers available with the new accelerators, the need for consistent and accurate covariant calculations will grow. The foundations for such calculations have been reviewed in this chapter.

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