BETATRON FUNCTION PARAMETERIZATION OF BEAM OPTICS INCLUDING ACCELERATION*

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I. Abstract

Betatron function parameterization of symplectic matrices is of recognized utility in beam optical computations. The traditional "beta functions" $\beta, \alpha, \gamma \; (= (1+\alpha^2)/\beta)$ and ψ (the betatron phase advance) provide an emittance-independent representation of the properties of a beam transport system. They thereby decouple the problem of "matching" injected beam envelope properties to the acceptance of a particular transport system from the details of producing a beam of a specific emittance. The definition and interpretation of these parameters becomes, however, more subtle when acceleration effects, especially adiabatic damping (with associated nonsymplecticity of the transfer matrix), are included.

We present algorithms relating symplectic representations of beam optics to the more commonly encountered nonsymplectic (x, x', y, y') representation which exhibits adiabatic damping. Betatron function parameterizations are made in both representations. Self-consistent physical interpretations of the betatron functions are given and applications to a standard beam transport program are made.

II. Canonical Formalism

In this paper, we will use a canonical formalism in which single-particle motion is described by a six-dimensional phase space $(x, p_x/p_0, y, p_y/p_0, t, E)$. Here, x and y are the usual perturbative transverse displacements from a design orbit, p_x and p_y the associated canonical momenta, and p_0 a reference momentum that is a constant usually chosen to be the injection momentum. The longitudinal variables t and E may be taken to be either the full time-of-flight and energy of a particle at a particular path length or infinitesimal deviations of these parameters from the values for a reference particle. The full linear dynamics of the system are then described by a group of 6×6 symplectic matrices[1].

We will restrict consideration to decoupled nondispersive systems. Transverse single particle dynamics are then described by 2×2 symplectic matrices. We consider the two-dimensional subspace $(x, p_x/p_0)$ in which the following statements apply to linear motion[2].

a. The beam phase space is parameterized by betatron functions $\beta, \alpha, \gamma \; (= (1+\alpha^2)/\beta)$, and ψ (betatron phase). At position s in the beam line, all particles in the beam lie within the ellipse

$$\gamma(s)x^2 + 2\alpha(s)x\frac{p_x}{p_0} + \beta(s)\left(\frac{p_x}{p_0}\right)^2 = \epsilon_0 \qquad (1)$$

where ϵ_0 is the beam emittance at the reference momentum p_0.

b. The emittance ϵ_0 is a constant of the motion (conserved at all s).

c. The transport system is described between positions s_1 and s_2 by the following betatron-function-parameterized symplectic matrix in which $\Delta\psi = \psi(s_2) - \psi(s_1), \beta_n = \beta(s_n)$ and $\alpha_n = \alpha(s_n)$,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

where

$$\begin{aligned} M_{11} &= \sqrt{\frac{\beta_2}{\beta_1}}(\cos\Delta\psi + \alpha_1 \sin\Delta\psi) \\ M_{12} &= \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ M_{21} &= -\frac{(1+\alpha_1\alpha_2)\sin\Delta\psi + (\alpha_2-\alpha_1)\cos\Delta\psi}{\sqrt{\beta_1\beta_2}} \\ M_{22} &= \sqrt{\frac{\beta_1}{\beta_2}}(\cos\Delta\psi - \alpha_2\sin\Delta\psi) \quad . \end{aligned} \qquad (2)$$

d. The betatron functions propagate between s_1 and s_2 according to the following relation, in which the M_{ij} are the elements of the above matrix.

$$\begin{pmatrix} \beta(s_2) \\ \alpha(s_2) \\ \gamma(s_2) \end{pmatrix} = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & 1+2M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \times \begin{pmatrix} \beta(s_1) \\ \alpha(s_1) \\ \gamma(s_1) \end{pmatrix} \qquad (3)$$

where

$$\tan(\psi(s_2) - \psi(s_1)) = \frac{M_{12}}{M_{11}\beta(s_1) - M_{12}\alpha(s_1)}$$

Use of the above formalism is contingent only upon the system being Hamiltonian. It therefore applies to transport systems including acceleration (providing synchrotron radiation is neglected), and provides the desired betatron function parameterization.

III. Betatron Parameterization of Noncanonical System

The traditional representation of optical systems is, however, not cast in the above canonical form. Instead, it is based on a phase space $(x, x', y, y', l, \delta)$, in which $x'(s) = p_x(s)/p(s)$ and $y'(s) = p_y(s)/p(s)$ are the ratios of the canonical transverse momenta to the full momentum at the point of observation. The resulting linear matrix representation of the motion is "adiabatically damped" if the transport system includes acceleration. Furthermore, the determinant of the matrix becomes the ratio of the injected momentum to the extracted momentum. As the matrix is not symplectic, it does not admit a representation in betatron functions (2).

It is, however, possible to recover a betatron function parameterization of the noncanonical $(x, x', ...)$ system by reference to the canonical $(x, p_x/p_0, ...)$ picture and use of the betatron function description of the canonical formalism. As above,

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we will restrict attention to decoupled nondispersive systems so that we need consider $(x, p_x/p_0)$ or (x, x') and 2×2 submatrices only. Denote by $M_{xp_x}(s_1, s_2)$ the symplectic matrix describing a transport channel from s_1 to s_2, and also denote by $M_{xx'}(s_1, s_2)$, the associated nonsymplectic matrix transforming the noncanonical variables (x, x') in the same system. At any location s, the canonical and noncanonical phase spaces are related via the following noncanonical transformation.

$$\begin{pmatrix} x \\ x' \end{pmatrix} = T(s) \begin{pmatrix} x \\ \frac{p_x}{p_0} \end{pmatrix} \tag{4}$$

Here $T(s)$ is a nonsymplectic matrix.

$$T(s) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{p_0}{p(s)} \end{pmatrix}$$

The symplectic and nonsymplectic transfer matrices are then readily related as follows, and as illustrated by Figure 1.

$$M_{xx'}(s_1, s_2) = T^{-1}(s_2) M_{xp_x}(s_1, s_2) T(s_1) \tag{5}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
x(s_1),\ x'(s_1) \arrow[r, "M_{x\,x'}(s_1{,}s_2)"] \arrow[d, "T(s_1)"'] & x(s_2),\ x'(s_2) \\
x(s_1),\ p_x(s_1) \arrow[r, "M_{x\,p_x}(s_1{,}s_2)"'] & x(s_2),\ p_x(s_2) \arrow[u, "T^{-1}(s_2)"']
\end{tikzcd}$$

Figure 1 Relation Between Canonical and Non-Canonical Representations

As M_{xp_x} can be parameterized in terms of betatron functions, and as $M_{xx'}$ can be related to M_{xp_x}, it is therefore reasonable to inquire as to the possibility of a parameterization of the (x, x') phase ellipse by means of betatron functions in a fashion similar to that employed in the $(x, p_x/p_0)$ phase ellipse.

We observe that at any displacement s along the beam line the beam is described in $(x, p_x/p_0)$ by (1). At the same location, the ellipse may be rewritten in the variables (x, x') through use of the transformation (4) given above. The result follows.

$$\overline{\gamma}(s)x^2 + 2\overline{\alpha}(s)xx' + \overline{\beta}(s)(x')^2 = \overline{\epsilon}(s) \tag{1'}$$

with

$$\begin{aligned}
\overline{\beta}(s) &= \beta(s)\frac{p(s)}{p_0} \\
\overline{\alpha}(s) &= \alpha(s) \\
\overline{\gamma}(s) &= \gamma\frac{p_0}{p(s)} \\
\overline{\epsilon}(s) &= \epsilon_0\frac{p_0}{p(s)} \\
\overline{\psi}(s) &= \psi(s)
\end{aligned} \tag{1''}$$

Equations $(1')$ and $(1'')$ provide the desired parameterization of the (x, x') representation in terms of betatron functions. In this parameterization, as in the canonical representation (1), the usual physical interpretations (based on the geometric characteristics of the phase ellipse) apply[3]. For example, the beam angular divergence at s is $\sqrt{\overline{\gamma}(s)\overline{\epsilon}(s)}$, while the beam spot size at s is $\sqrt{\beta(s)\epsilon_0} = \sqrt{\overline{\beta}(s)\overline{\epsilon}(s)}$. The final equality of equation $(1'')$ is by virtue of the fact that the betatron phase (equivalent to the number of betatron oscillations executed by a particle) must be independent of the representation chosen to describe the motion. The particles don't care how we describe them; they oscillate the same number of times in either case.

IV. Transport of Betatron Functions

In the case of the canonical formalism, equations (2) and (3) specify both the single particle linear dynamics and the propagation of the phase ellipse parameters. In the noncanonical (x, x') formalism, the matrix $M_{xx'}$ specifies the linear single particle dynamics, but as $M_{xx'}$ is nonsymplectic, it does not admit the particular representation (2) in terms of betatron functions. Hence, direct application of (3) to $M_{xx'}$ for the purpose of betatron function propagation is not possible. The (x, x') phase ellipse parameterization $(1')$ and $(1'')$ does, however, allow an indirect method of phase ellipse parameter propagation in the noncanonical case. The propagation of "noncanonical" betatron functions $\overline{\beta}, \overline{\alpha}, \overline{\epsilon}$, and $\overline{\psi}$ is accomplished by transformation to the canonical picture, propagation of the betatron functions within the canonical representation, and subsequent reconversion to the noncanonical description.

We therefore outline the solution to the problem of propagation of noncanonical betatron functions: given x, x' betatron functions $\overline{\beta}_1, \overline{\alpha}_1, \overline{\epsilon}_1$, and $\overline{\psi}_1$, at position s_1, and a transfer matrix $M_{xx'}(s_1, s_2)$ describing a beamline from s_1 to s_2 in the noncanonical representation, how are the betatron functions $\overline{\beta}_2, \overline{\alpha}_2, \overline{\epsilon}_2$, and $\overline{\psi}_2$, at position s_2 specified?

a. Generate canonical betatron functions $\beta_1, \alpha_1, \gamma_1$, and ϵ_0 at s_1 using the inverse of the transformation $(1'')$:

$$\begin{aligned}
\beta_1 &= \overline{\beta}_1\frac{p_0}{p(s_1)} \\
\alpha_1 &= \overline{\alpha}_1 \\
\gamma_1 &= \overline{\gamma}_1\frac{p(s_1)}{p_0} \\
\epsilon_0 &= \overline{\epsilon}_1\frac{p(s_1)}{p_0} \\
\psi_1 &= \overline{\psi}_1
\end{aligned}$$

This transformation may be represented by equation (3) through use of the matrix

$$\mathcal{T}(s_1) = \begin{pmatrix} \sqrt{\frac{p_0}{p(s_1)}} & 0 \\ 0 & \sqrt{\frac{p(s_1)}{p_0}} \end{pmatrix} \tag{6}$$

We remark that the phase advance is invarient, as it must be, under this change of representation: $\overline{\psi}(s_1) = \psi(s_1)$. That is, the number of betatron oscillations excuted by a particle is independent of representation.

b. Construct from $M_{xx'}(s_1, s_2)$ the symplectic matrix $M_{xp_x}(s_1, s_2)$ propagating $(x, p_x/p_0)$ and the canonical betatron functions from s_1 to s_2. This is done by use of equation (5).

$$M_{xp_x}(s_1, s_2) = T(s_2) M_{xx'} T^{-1}(s_1) \tag{5'}$$

c. Propagate the canonical betatron functions from s_1 to s_2 using the matrix $M_{xp_x}(s_1, s_2)$ constructed in $(5')$.

d. Transform the canonical betatron functions $\beta_2, \ldots$ to the noncanonical representation $\overline{\beta}_2, \ldots$ using the mapping ($1''$). Observe that this is the inverse of the transformation performed in item a.; the transformation may be accomplished through the use of (3) and the matrix representation of $\mathcal{T}^{-1}(s_2)$, where $\mathcal{T}(s)$ is defined as in equation (6).

We remark that in all of the above transformations, the phase advances $\psi(s)$ and $\overline{\psi}(s)$ have the same value and interpretation in either the canonical or the noncanonical representations. This is, as noted before, because the number of betatron oscillations executed by a particle is a physical observable which must be independent of choice of representation.

The sequence of tranformations a. – d. may be compressed to a single operation through concatenation of the various matrices involved. Specifically, operations a through d are equivalent to a single operation, that of propagation of the noncanonical betatron functions from s_1 to s_2 with (3) using the symplectic 2×2 matrix $\mathcal{M}(s_1, s_2)$ defined as follows.

$$\begin{aligned}\mathcal{M}(s_1,s_2) &= \mathcal{T}^{-1}(s_2)T(s_2)M_{xx'}(s_1,s_2)T^{-1}(s_1)\mathcal{T}(s_1)\\ &= \begin{pmatrix}\sqrt{\frac{p(s_2)}{p_0}} & 0\\ 0 & \sqrt{\frac{p(s_2)}{p_0}}\end{pmatrix} \quad M_{xx'}(s_1,s_2)\\ &\qquad\qquad \times \begin{pmatrix}\sqrt{\frac{p_0}{p(s_1)}} & 0\\ 0 & \sqrt{\frac{p_0}{p(s_1)}}\end{pmatrix}\end{aligned} \tag{7}$$

It then follows that

$$\mathcal{M} = \begin{pmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}$$

with

$$\begin{aligned}\mathcal{M}_{11} &= \sqrt{\frac{\overline{\beta}_2}{\overline{\beta}_1}}(\cos\Delta\overline{\psi} + \overline{\alpha}_1\sin\Delta\overline{\psi})\\ \mathcal{M}_{12} &= \sqrt{\overline{\beta}_1\overline{\beta}_2}\sin\Delta\overline{\psi}\\ \mathcal{M}_{21} &= -\frac{(1+\overline{\alpha}_1\overline{\alpha}_2)\sin\Delta\overline{\psi} + (\overline{\alpha}_2 - \overline{\alpha}_1)\cos\Delta\overline{\psi}}{\sqrt{\overline{\beta}_1\overline{\beta}_2}}\\ \mathcal{M}_{22} &= \sqrt{\frac{\overline{\beta}_1}{\overline{\beta}_2}}(\cos\Delta\overline{\psi} - \overline{\alpha}_2\sin\Delta\overline{\psi})\end{aligned} \tag{2'}$$

The noncanonical betatron functions propagate formally as in equation (3), using $\mathcal{M}$ instead of M_{xp_x}, and using the noncanonical betatron functions $\overline{\beta}, \overline{\alpha}, \overline{\psi}$ instead of the canonical functions $\beta, \alpha, \overline{\psi}$.

The symplecticity of $\mathcal{M}$ stems from the fact that, by construction (equation (7)), it is a 2×2 unimodular matrix. We emphasize that $\mathcal{M}$ harbors physical significance only by virtue of the fact that it is a symplectic matrix parameterized (equation (2)) by the noncanonical betatron functions, and that it therefore serves to propagate them from s_1 to s_2. It **does not** describe the dynamics of the system in the (x, x') representation. Thus, for example, ray-tracing calculations should be carried out using the matrix $M_{xx'}$, **not** the matrix $\mathcal{M}$.

V. Example

Application of the above formalism has been made during the design of the CEBAF recirculating linac[4]. Computation of beam envelope functions including acceleration effects was required to allow matching of the multiple recirculated from linac to recirculation transport lines. The numerical calculations were performed using a specially modified version of the program DIMAD[5], in which a simple matrix model of the effect of an RF cavity with finite energy gain upon the (x, x') phase space was implemented. Using the "gmatrix" element input feature of the program, the matrices M_{xp_x} and $\mathcal{M}$ were calculated between various points in the machine, and both canonical and noncanonical betatron functions were propagated through the transport system.

The results of these calculations are illustrated in Figures 2 and 3. Figure 2 displays the canonical beta function β_x through the first pass of the linac portion of the accelerator. In this case, the beam spot size at any position s is given by $\sqrt{\beta_x(s)\epsilon_0}$, with ϵ_0 being the injected emittance. Figure 3 displays the noncanonical betatron function $\overline{\beta}_x$ in the same region of the machine. Here, the beam spot size at any location is taken to be $\sqrt{\overline{\beta}_x(s)\overline{\epsilon}(s)}$, with $\overline{\epsilon}(s)$ the adiabatically damped emittance at position s:

$$\overline{\epsilon}(s) = \epsilon_0 \frac{p_0}{p(s)}.$$

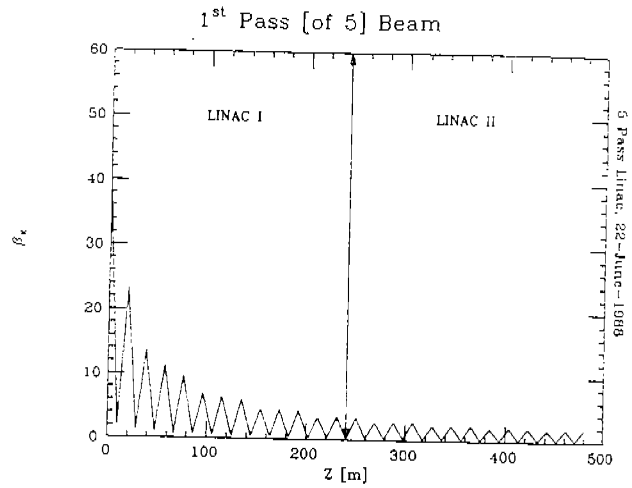

Figure 2 Transport of Canonical Beta Function

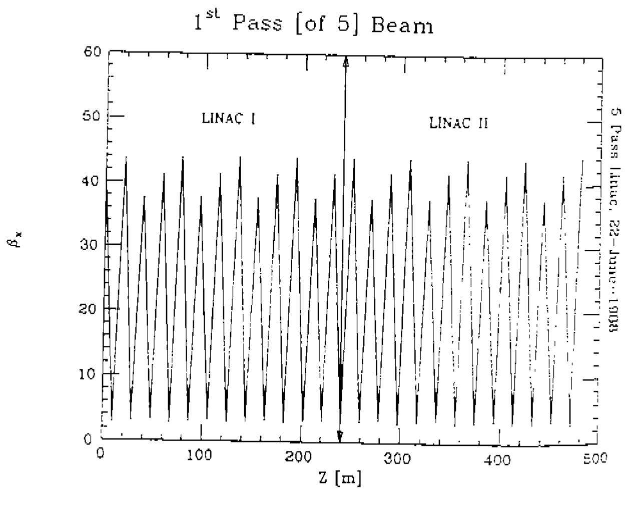

Figure 3 Transport of Non-Canonical Beta Function

VI. Acknowledgement

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VII. References

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