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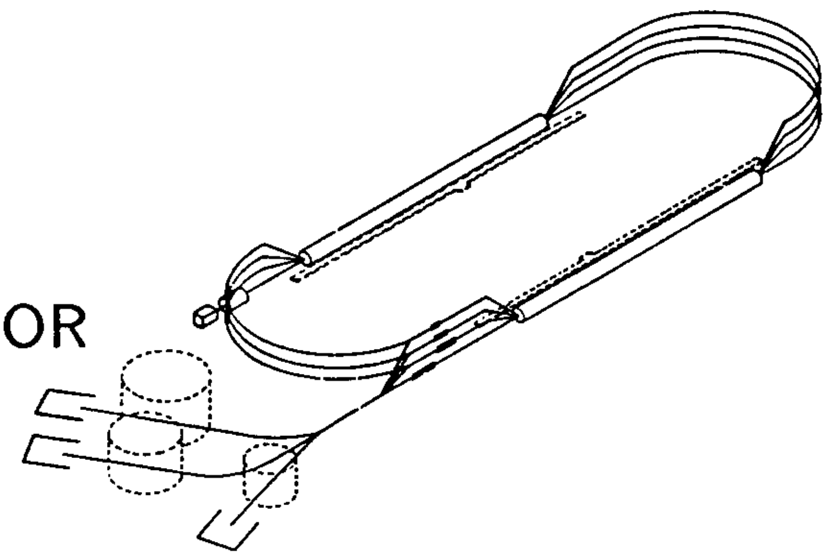

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An iterative method for calculation of the narrow-band impedance is described for a system with a small variation in boundary conditions, so that the variation can be considered as a perturbation. The results are compared with numeric calculations. The method is used to relate the origin of the trapped modes with the degeneracy of the spectrum of an unperturbed system. The method also can be applied to transverse impedance calculations.

1. INTRODUCTION

In many cases a system can be described as a combination of simple systems, each of which can be relatively easily described analytically. For example, we have a pill-box cavity with narrow tubes attached to it. This system may not have cylindrical symmetry, but the field pattern can be close to that of the closed pill-box cavity almost everywhere. Another example of system that could be considered similarly are weakly coupled cavities.

In this paper we describe a perturbation theory that gives expression to the field pattern, the impedance, the width of the narrow-band impedance, and the shift of frequencies, all in terms of the parameters for an "ideal" unperturbed system, for which the eigen modes and frequencies are known. We prefer to describe the method on simple pill-box cavity with attached tubes, rather than giving a general formulation of the approach. This clarifies the idea and allows comparison of the result with numeric calculations. The general use of the method and its applicability are expected to be clear from the example.

In the last section we use the method to explain the existance of the so-called trapped modes, i.e., very narrow resonances substantially above cut-off that were found in numeric simulations[1]. We relate the origin of the trapped modes to the approximate degeneracy of the spectrum of the "ideal system".

2. FORMULATION OF THE BOUNDARY CONDITIONS

First, let us formulate the exact boundary conditions for the system. The pill-box cavity has width g and radius b; attached tubes with radii a have their axes in the direction of the z-axis with the opening at $|z| > g/2$. The tangential electric field on the boundary is zero everywhere except at the openings, where the tangential component can be related with the normal component.

The field in the tube is the superposition of the waves $\psi_m(r, z)$:

$$E_z = \sum_m \frac{\nu_m}{a} B_m \psi_m(r, z)$$

and (1)

$$E_r = \sum_m i\lambda_{am} B_m \frac{\partial \psi_m}{\partial r}$$

where

$$\psi_m(r, z) = \frac{1}{N_m} (\frac{\nu_m}{a}) J_0 (\frac{\nu_m r}{a}) e^{i\lambda_{am} z} ,$$

$$\lambda_{am} = \sqrt{k^2 - (\frac{\nu_m}{a})^2}, \; k = \frac{\omega}{c}, \; J_0(\nu_m) = 0\,,$$

$$N_m = \frac{\nu_m}{\sqrt{2}} J_1(\nu_m)\,,$$

$$< m|m' > \equiv \int_0^a \psi_m^* \psi_n r dr = \delta_{m,n}\,.$$

The coefficient B_m of Eq. (1) can be found from the continuity of the normal component of the field E_z at the opening as

$$B_m = \frac{a}{\nu_m N_m} < \psi_m | E_z(r, g/2) > \tag{2}$$

B_m defines the tangential component of the field in the tube and, from the continuity, the tangential component of the field in the cavity. The field component is equal to zero everywhere on the surface of the cavity except at the openings, where it must satisfy the condition:

$$E_r = \sum_m \frac{ia\lambda_{am}}{2\pi\nu_m} \frac{\partial \psi_m}{\partial r} \int dS \psi_m^* E_z. \tag{3}$$

Here the integral and the functions ψ_m,E_r and E_z are taken on the surface S of the openings.

Eq. (3) is the exact boundary condition that relates the r and z components of the field *in* the cavity at the openings. Subsequently, we can forget the tubes and consider the wave equation within the cavity only. The equation however has to be solved with the boundary condition of Eq. (3) instead of setting the tangential component of the field to zero as in the case of a closed cavity.

Eq. (3) gives the linear relationship between the tangential components of the electric field and the magnetic field H_ϕ, where $H_\phi = \frac{i}{k}\partial E_z/\partial r$ in the form

$$E_r = \varsigma H_\phi.$$

The coefficient ς, as defined, is the effective surface impedance of the opening. Hence, the tune shift, caused by the opening, can be calculated using the well known result[2] for the tune shift due to surface impedance:

$$\omega - \omega_0 = -\frac{ic}{2}\frac{\int dS\varsigma|\vec{H}|^2}{\int(|\vec{H}|^2 - |\vec{E}|^2)(\vec{r}d\vec{S})}. \tag{4}$$

3. PERTURBATION THEORY, MODAL ANALYSIS.

The Fourier component of the EM field in a cavity with frequency ω, excited by an ultrarelativistic particle with charge Q, can be written as

$$E_r = \gamma M G_1(r,b)e^{ikz} + \frac{2iQa}{\pi c}\sum_n \frac{\nu_n}{b}\lambda_{bn}J_1(\frac{\nu_n r}{b})d_n^{\pm}\begin{pmatrix} i\sin\lambda_{bn}z \\ \cos\lambda_{bn}z \end{pmatrix} \tag{5}$$

and

$$E_z = -iMG_0(r,b)e^{ikz} + \frac{2iQa}{\pi c}\sum_n (\frac{\nu_n}{b})^2 J_0(\frac{\nu_n r}{b})d_n^{\pm}\begin{pmatrix} \cos\lambda_{bn}z \\ i\sin\lambda_{bn}z \end{pmatrix} \tag{6}$$

where $d_n^{\pm}$ are the amplitudes of the even and odd modes respectively, and

$$k = \frac{\omega}{c} \quad \text{and} \quad M = \frac{Qk}{\pi c\gamma^2}.$$

The cylindrical coordinate system is choosen with the axis along that of the pill-box cavity, and $z = \pm\frac{g}{2}$ at the openings, b is the radius of the cavity, and g is the width of it. The functions G_0 and G_1 in Eqs. (5) and (6) are

$$G_0(r,b) = K_0(\frac{kr}{\gamma}) - I_0(\frac{kr}{\gamma})\frac{K_0(\frac{kb}{\gamma})}{I_0(\frac{kb}{\gamma})}$$

and

$$G_1(r,b) = K_1(\frac{kr}{\gamma}) - I_1(\frac{kr}{\gamma})\frac{K_0(\frac{kb}{\gamma})}{I_0(\frac{kb}{\gamma})}.$$

$K_{0,1}$ are McDonald functions. For $\gamma >> 1$,

$$\gamma M G_1(b,b) = \frac{Z_o Q}{4\pi^2 b}, \quad Z_o = 377\Omega.$$

The substitution of Eqs. (5) and (6) into Eq. (3) gives a set of linear equations for amplitudes $d^{\pm}$:

$$d_n^+ = \frac{i}{\sin\chi_n}[p_n \sin\mu - \frac{1}{2}\sum_m \Gamma_{m,n} d_m^+ \cos\chi_m] \tag{7}$$

and

$$d_n^- = \frac{i}{\cos\chi_n}[p_n \cos\mu + \frac{1}{2}\sum_m \Gamma_{m,n} d_m^- \sin\chi_m] \tag{8}$$

where

$$\chi_m = \frac{g}{2}\lambda_{b,m}, \quad \mu = gk/2.$$

The parameters $\Gamma_{n,m}$ and p_n are defined as follows:

$$\Gamma_{n,m} = \frac{4g\nu_m^2}{a\chi_m}(\frac{a}{b})^4 \frac{J_0(\nu_n a/b) J_0(\nu_m a/b)}{J_1^2(\nu_m)}\sigma_{n,m} \tag{9}$$

where

$$\sigma_{n,m} = \sum_l \frac{\sqrt{(ka)^2 - \nu_l^2}}{[\nu_l^2 - (\nu_n a/b)^2][\nu_l^2 - (\nu_m a/b)^2]},$$

and

$$p_n = (g/2a\chi_n)J_0(\nu_n a/b)/J_1^2(\nu_n)$$

Substitution Eqs. (5) and (6) into the definition of the impedance,

$$Z(\omega) = -\frac{2\pi}{Q}\int_{-g/2}^{g/2} dz E_z(a,z)e^{-ikz}$$

gives

$$Z(\omega) = \frac{Z_0}{\pi}\frac{g}{2b}(\frac{a}{b})\sum_n J_0(\nu_n a/b)[(d_n^+ + d_n^-)\frac{\sin(\mu-\chi_n)}{(\mu-\chi_n)} + (d_n^+ - d_n^-)\frac{\sin(\mu+\chi_n)}{(\mu+\chi_n)}] \quad (10)$$

A set of equations equivalent to Eqs. (7) and (8) can be obtained by matching the fields in the regions $r < a$ and $r > a$ at $r = a$ [3]. Both methods of matching are equivalent.

The Eqs. (7) and (8) were solved by truncation so that only a finite number of equations was retained [4,5]. The accuracy of the truncated system depends both on the rank of the matrix retained and on the method of matching. For small openings calculation of narrow-band impedance by matching at $z = \pm g/2$ is preferable, while for broad-band impedance matching at $r = a$ is adequate.

In this paper we calculate the narrow-band impedance. It can be expected that for a narrow opening, the field pattern is close to that in the closed cavity. Therefore, for such frequencies that the following conditions are met

$$\sin\chi_n \approx 0, \quad \chi_n \approx \chi_n^o = n\pi \quad \text{odd} \quad \text{modes},$$

and

$$\cos\chi_n \approx 0, \quad \chi_n \approx \chi_n^o = (n+1/2)\pi \quad \text{even} \quad \text{modes},$$

only modes $d_n^{\pm}$ may be retained in the Eqs. (7) and (8). This gives

$$d_n^+ = \frac{ip_n \sin\mu}{\sin\chi_n + \frac{i}{2}\Gamma_{nn}\cos\chi_n} \tag{11}$$

and

$$d_n^- = \frac{ip_n \cos\mu}{\cos\chi_n - \frac{i}{2}\Gamma_{nn}\sin\chi_n} \tag{12}$$

Other amplitudes d_m, where $m \neq n$, describe the mixing of the modes of the closed cavity and in this approximation are zero. The next iteration, substitution of Eqs. (11) and (12) into the right-hand side of Eqs. (7) and (8), gives $d_m \neq 0$. The iteration then can be repeated.

Eqs. (11) and (12) have a typical Breit-Wigner resonance structure with the width γ_{nm} of the resonance lines

$$\gamma_{n,n} = \frac{4\chi_n^o}{g^2 k_n}\Gamma_{n,n} \tag{13}$$

This expression is simply the ratio of the energy leak in the tubes $\dot{W}_n$ to the energy of the mode W_n:

$$\gamma_{n,n} = -\frac{\dot{W}_n}{W_n}. \tag{14}$$

The energy flow $\dot{W}_{n,i}$ in the i- mode of the tube field with amplitude $B_{n,i}$, excited by coupling with the n- mode of the field in the cavity, is given by the integral of the Poynting vector over the cross-section of the tube. The result is

$$\dot{W}_{n,i} = -\frac{ck}{4}\lambda_{a,i}|B_{n,i}|^2\nu_i^2 J_1^2(\nu_i).$$

The coefficients $B_{n,i}$, and hence the energy of the n- mode W_n, can be calculated according to Eq. (2)

$$B_{n,i} = \frac{2}{\nu_i^2 J_1^2(\nu_i)} e^{i\lambda_{a,i}g/2} \int_0^a r dr J_0\left(\frac{\nu_i r}{a}\right) E_{z,n}^o(r, g/2) \tag{15}$$

if the field $E_{z,n}^o$ in the cavity is approximated by the field of the n- mode of the closed pill-box cavity. This approximation gives the same result, as Eq. (13).

cavities can cause a multipole reflection of the wave and, as a result, give a long decay time to the mode. This explanation does not seem to be satisfactory because, above cut-off, the reflection rate is relatively small even on sharp edges and goes to zero rapidly if the edges are rounded. The rate is exponentially small if the function, describing the boundary, is continuous with all its derivatives.

Another explanation is that each mode in the cavity generates a wave in the tube that, under certain conditions, can cancel one another. We considered this explanation for the trapped mode of the pill-box cavity with attached tubes that can be seen in Fig. 2 as the small spike near $ka \approx 4.5$. The amplitude of the spike is actually much higher then shown in Fig. 2, and it gets bigger if the simulation is done with smaller steps of ka.

Calculations with URMEL confirmed existence of the trappped mode. In Table 1 below some frequencies f and ratios of shunt impedances to Q-factors r/Q are shown. They were found by URMEL for pill-box cavities with attached tubes closed at the ends with different ratios of the total half-lengths of the structure to the tube radius l/a. Other parameters are $a/b = 0.318$; $g/2b = 0.600$.

Table 1. Frequency and r/q

$l/a = 0.0$		$l/a = 5.0$		$l/a = 8.0$	
f	r/Q	f	r/Q	f	r/Q
218.276	1.50	223.534	30.65	219.533	2.10
232.173	0.32	232.660	0.45	no mode found	
232.485	0.40	233.338	0.00	233.236	0.015
234.881	0.14	239.157	4.50	239.296	3.06

For the mode in the third line with $f \approx 233.3$, which corresponds to the mode $ka \approx 4.5$, the field goes rapidly to zero outside of the cavity, and this corresponds to the definition of trapped modes. The ratio r/Q for this mode is unusally small.

If $ka = 4.5$, there is only a single wave that can propagate in the tube. In the closed pill-box cavity with the parameters given in Fig. 1, there are two modes of the same parity and with field pattern of the following form:

$$E^z_{\nu,l} \sim J_0(\nu r/b)\cos(l\pi z/g),$$

The modes have eigen frequencies close to $ka = 4.5$; (1) at $k_{\nu_2,l}a = 4.5176$, $\nu_2 = 5.5200$, and $l = 5$ and (2) at $k_{\nu_4,l}a = 4.5053$, $\nu_4 = 11.7915$ and $l = 3$. Fig. 4 shows the spectrum of the closed pill-box cavity under consideration. The wave in the tube is mostly generated by coupling with the two resonant modes in the cavity. The amplitudes $B_{n,1}$ of the wave in the tube can be calculated according to Eq. (15), where $E_{z,n}$ is defined by Eq. (6) with the coefficient d_n^- given in Eq. (12). The first term in Eq. (6) is proportional to $1/\gamma^2$ and negligibly small at high energies. Omitting the common factor in both modes, the amplitudes $B_{n,1}$ can be written as

$$B_{n,1} \propto \quad \frac{\nu_n^2 p_n J_0(\nu_n a/b)}{\nu_1^2 - (a\nu_n/b)^2} \quad \frac{\sin\chi_n}{\cos\chi_n - \frac{i}{2}\Gamma_{nn}\sin\chi_n} \tag{17}$$

where the p_n and Γ_{nn} are defined in Eq. (9).

Numeric calculation gives the values for them:

$$p_2 = 0.760 \quad \text{and} \quad \Gamma_{2,2} = 0.363 \quad \text{for} \quad \text{mode} \quad \nu_2 = 5.520,$$

$$p_4 = -2.96 \quad \text{and} \quad \Gamma_{4,4} = 0.753 \quad \text{for} \quad \text{mode} \quad \nu_4 = 11.792.$$

These values give the ratio of the amplitudes

$$\frac{B_{4,1}}{B_{2,1}} = -3.07$$

Therefore the trapped mode can not be explained by destructive interference in a straightforward sense, at least for this example.

We believe that the existence of two almost exactly degenerate modes in the cavity is crucial and might provide another explanation for the origin of the trapped modes. The two degenerate modes cannot be considered independently. Neglecting the amplitudes in Eqs. (7) and (8) except those of the two resonance modes, the system is reduced to two equations:

$$d_2 = \frac{i}{\cos\chi_2}[p_2 \cos\mu + \frac{\Gamma_{22}}{2} d_2 \sin\chi_2 + \frac{\Gamma_{2,4}}{2} d_4 \sin\chi_4]$$

and (18)

$$d_4 = \frac{i}{\cos\chi_4}[p_4 \cos\mu + \frac{\Gamma_{42}}{2} d_2 \sin\chi_2 + \frac{\Gamma_{4,4}}{2} d_4 \sin\chi_4].$$

Usually, the widths of the resonances are of the order of Γ_{22} and Γ_{44} and are relatively large for $ka \approx 4.5$. In the ka interval $4.3 - 4.8$, the range of variation of the coefficients of the Eq. (16) is relatively small, and the coefficients never go through zero. The range of the coefficients are:

p_1	$(0.805) - (0.709)$,	p_2	$(-3.5) - (-2.49)$,
Γ_{22}	$(0.358 + i5.510^{-3})$	$-$	$(0.367 + i4.210^{-3})$,
$\Gamma_{4,4}$	$(0.838 + i0.26)$	$-$	$(0.688 + i0.15)$,
$\Gamma_{2,4}$	$(2.34 - i0.16)$	$-$	$(1.92 - i0.096)$,
$\Gamma_{4,2}$	$(0.128 - i0.8810^{-2})$	$-$	$(0.131 - i0.6510^{-2})$

The determinant of the system of Eq. (18), however, changes substantially in this interval, and crosses zero as shown in Fig. 5. At the zero point the two modes of the cavity are mixed strongly and could give one mode that is completely decoupled from the field in the tubes and corresponds to the trapped mode.

In Fig. 6 the frequencies of the closed pill-box cavity are plotted vs the ratio of the radii a/b with a fixed value of g/2b = 0.600. Under these conditions, two frequencies remain degenerate. The trapped mode exists for all a/b and it location, indicated by the crosses, follows the variation of the degenerate frequencies exactly.

It would be interesting to determine, how general this situation is.

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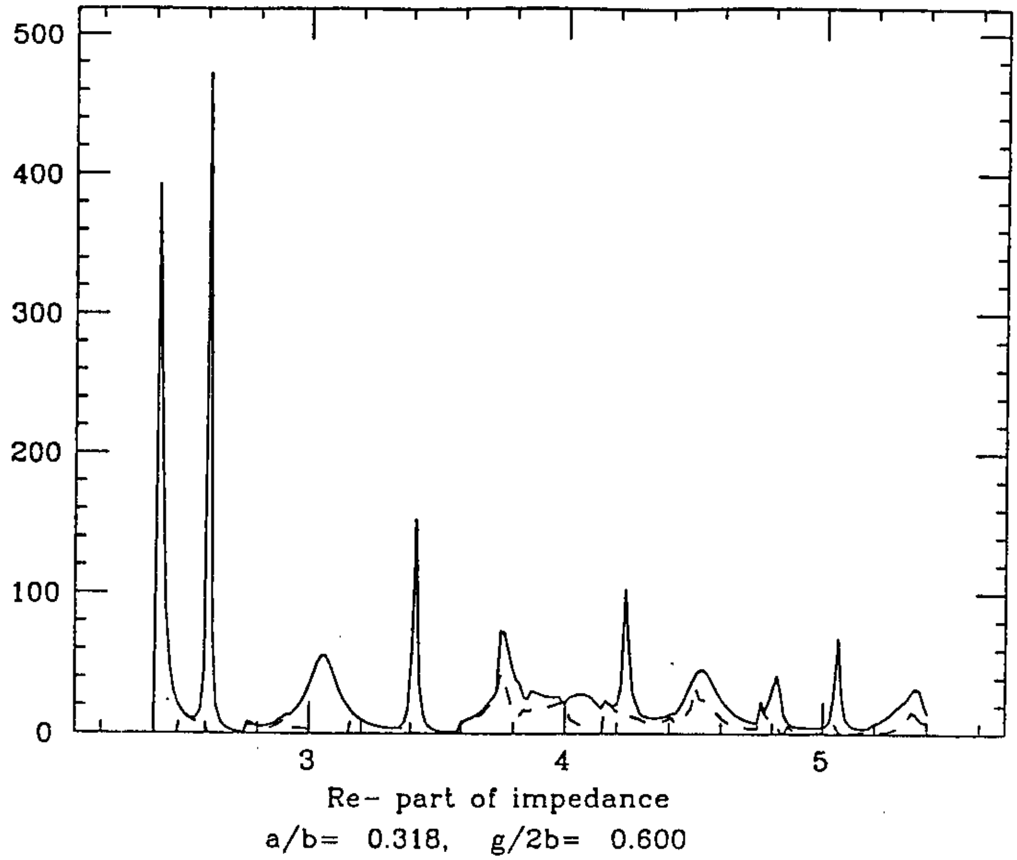


FIGURE 1a Real part of the impedance, - the single mode closest to ka is taken into account.

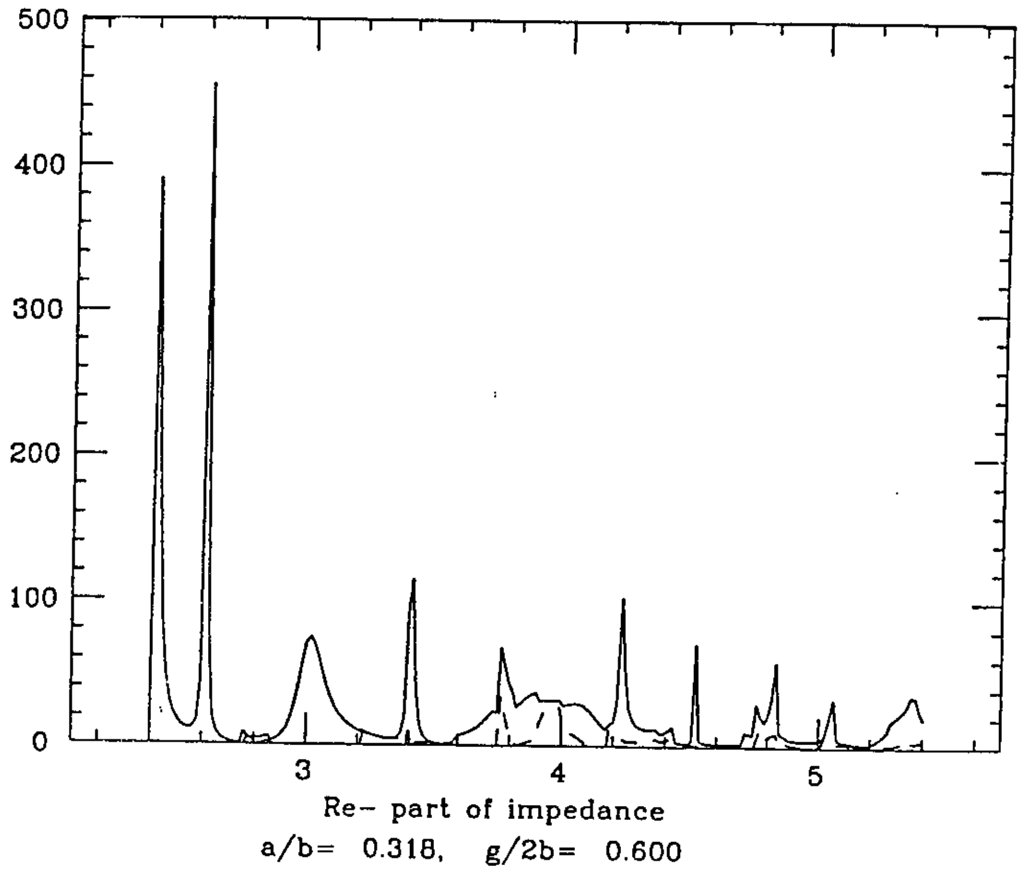


FIGURE 1b The same as Fig. 1, but the two closest to ka modes are taken into account.

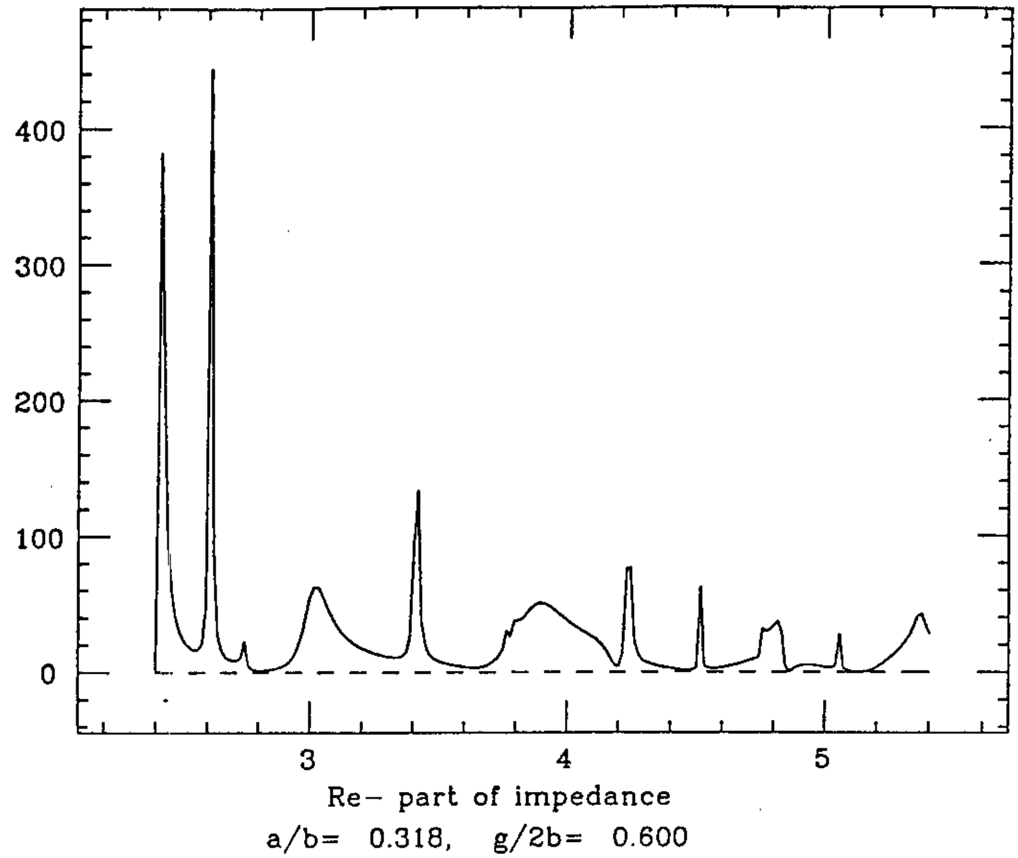


FIGURE 2 Real part of the impedance given by the truncated system of equations.

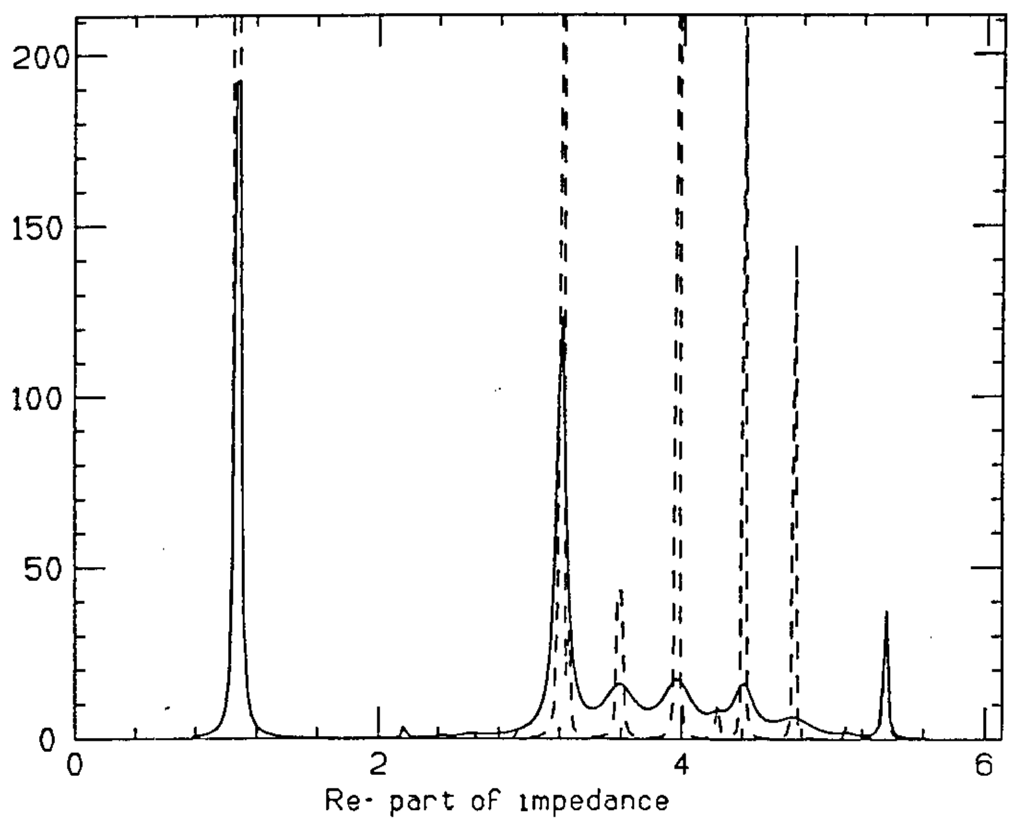


FIGURE 3 Real part of the impedance for CEBAF's cavity.

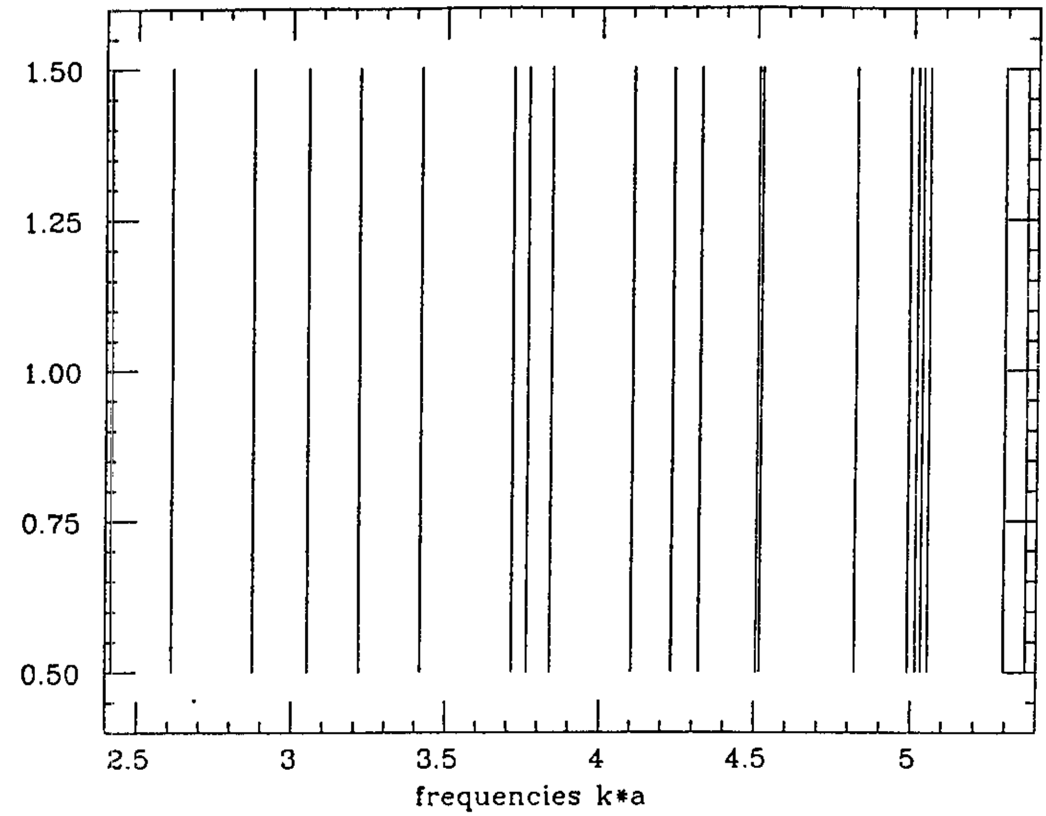


FIGURE 4 The spectrum of the closed pill-box cavity.

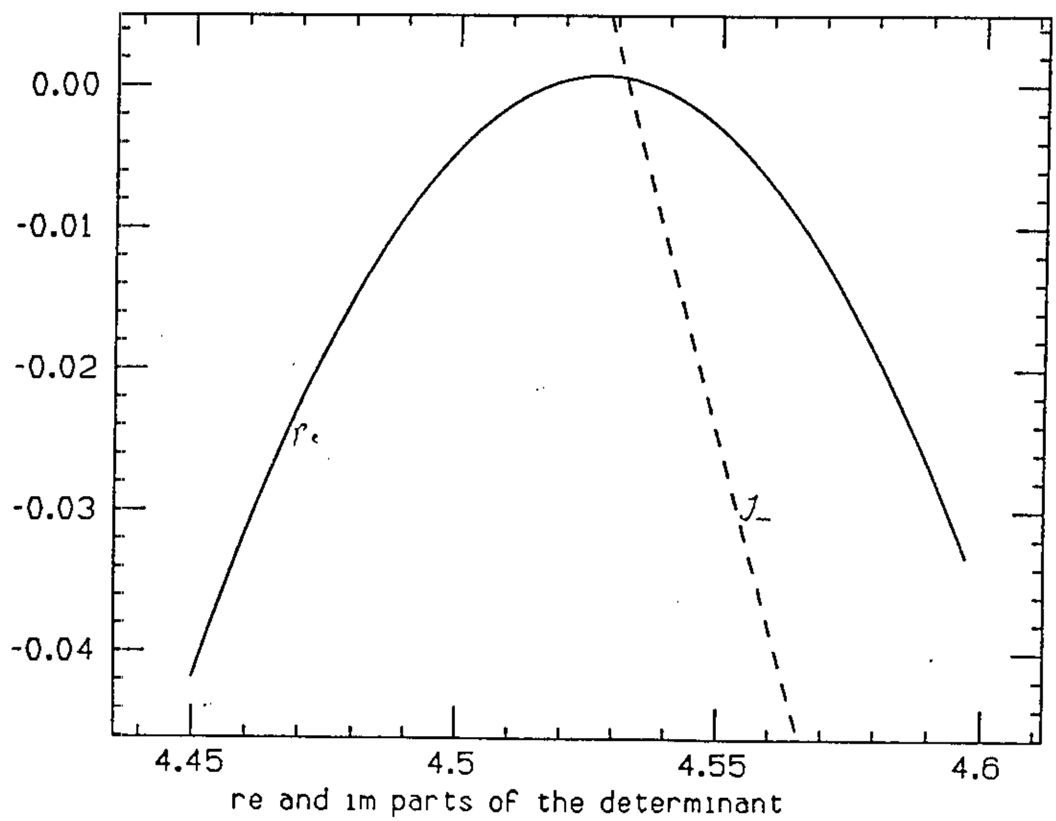
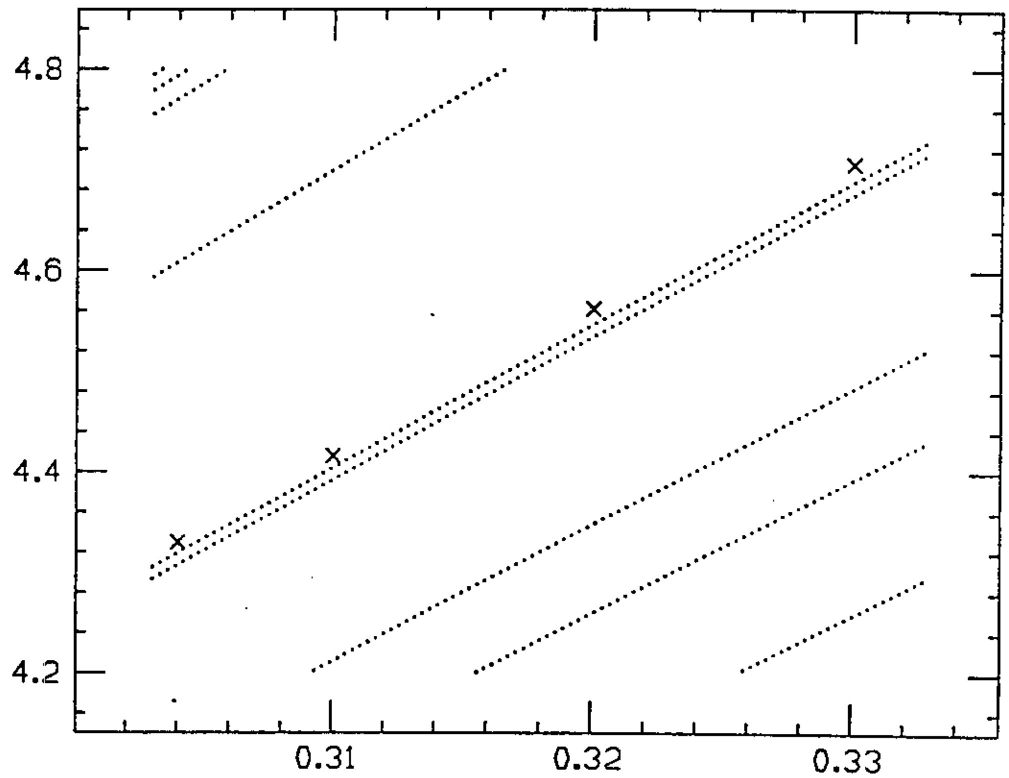


FIGURE 5 The real and imaginary parts of the determinant for the system (18) vs ka.



Frequencies vs. a/b. Crosses indicate the location of the trapped mode

FIGURE 6 The frequencies and locations of the trapped mode as a function of a/b at $g/2b = 0.600$.