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RELATIVISTIC EQUATIONS

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Relativistic Equations*

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In the last 3 years I have given several talks on relativistic equations for nuclear physics, and on relativistic effects in few nucleon systems[1]. In preparing for these talks, and in responding to the discussion which followed them, my thinking about relativistic equations and relativistic effects has gradually evolved. I want to begin this talk by focusing on several issues which naturally arise when discussing this subject, and then turn to a brief discussion of relativistic three nucleon equations. The latter topic has not been reviewed recently, and is particularly of current interest because of the beautiful form factor measurements of the ^{3}He - ^{3}H system recently completed at Saclay[2] and Bates[3], and the accompanying speculation that relativistic effects are important for understanding the three-nucleon system.

I. Issues

Relativistic equations for the two body scattering amplitude take the following very general form

$$M = V + V\,G\,M \tag{1}$$

where M is the 2 body scattering matrix, V the relativistic kernel or potential and G the two body propagator. Equation (1) can be regarded as a shorthand for the infinite sum

$$\begin{aligned} M &= V + V\,G\,V + V\,G\,V\,G\,V + V\,G\,V\,G\,V\,G\,V + \ldots. \\ &= [1 - V\,G]^{-1}V \end{aligned} \tag{2}$$

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where the n^{th} term in the sum is the n^{th} Born approximation to the amplitude. Solving Eq. (1) is a method of summing the series (2) which is essential when many terms contribute to the series (which usually is the case for low energy scattering) or when the series diverges (near the bound state poles of M).

1.1 Lorentz invariance

The first issue concerns the meaning of "relativistic" in the content of this subject. What I shall assume is that the Eq. (1) and the corresponding series (2) are Lorentz invariant in the sense that (a) it is clear from the equations themselves how to calculate M in any frame, that (b) a law telling how to transform M from one frame to another can be deduced, and that this law makes it possible to write the components of M in any frame in terms of those in, say, the rest frame, so that M need be calculated originally only in one frame, and (c) matrix elements involving M can be explicitly shown to be Lorentz invariant using the transformation law for M.

Many equations which use relativistic kinematics or are derived from relativistic objects (such as Dirac spinors) do not satisfy these stringent requirements. For example, Bag models which are based on a relativistic lagrangian density are not Lorentz invariant because we do not yet know how to boost bag states, although some progress on this topic has been made recently. For the same reason, wave equations based on time-ordered perturbation theory which use relativistic kinematics may also not satisfy these criteria, even though they may provide an excellent dynamical description of the nucleon-nucleon interactions, and include some effects of relativistic origin[4].

It is important to compare intermediate energy data with Lorentz invariant calculations -- only in this way can we eliminate uncertainties arising from the breaking of Lorentz invariance and assure ourselves that we are really testing the underlying dynamics. This is particularly important for programs which study the few body system at intermediate energies, such as those which we expect to carry out at CEBAF.

1.2 Choice of propagator

The requirement of Lorentz invariance does not uniquely define the equation; it is necessary to specify the propagator. A large number of choices are possible, but three which have received the most attention are the Bethe-Salpeter (BS)[5], the one-particle-on-shell equation (G_1)[6], and the Light Front (LF)[7]. If two particles have total 4-momentum P and relative 4-momentum p, so that

$$\begin{aligned} P &= p_1 + p_2 \\ p &= \frac{1}{2}\,(p_1 - p_2) \end{aligned} \tag{3}$$

then in all cases the total momentum is conserved, and the propagator depends on p only. In the BS case it depends on all 4 components of p, so that for spin zero particles the BS propagator is

$$\int d^4p\ G_{BS}(p) = \int d^4p\ [m_1^2 - (\tfrac{1}{2}P + p)^2]^{-1}[m_2^2 - (\tfrac{1}{2}\,P - p)^2]^{-1} \tag{4}$$

where m_1, and m_2 are the masses of the two particles. For the one particle on shell equation (particle 2 if $m_2 > m_1$), the propagator depends only on three components of p, the 4^{th} being constrained in a Lorentz invariant manner by the mass-shell condition, so that

$$\begin{aligned} \int d^4p\ \delta_+[m_2^2 - (\tfrac{1}{2}\,P - p)^2]\ G_1(p) &= \int d^4p_2\delta_+[m_2^2 - p_2^2]\ [m_1^2 - (P - p_2)^2] \\ &= \int \frac{d^3p}{2E_2}\ \ [E_1^2 - (W-E_2)^2]^{-1} \end{aligned} \tag{5}$$

where the last expression holds in the enter of mass of the pair, with $P = (W,\vec{0})$ and $E_{1,2} = (m_{1,2}^2 + \vec{p}^{\,2})^{1/2}$. (It is important to identify the δ function with the volume integral -- particle 2 in this description is removed from the equation from the start and does not propagate.) One may also specify particle one to be on shell. Finally, in the LF formalism, the light front variables $p^{\pm} = p_o \pm p_3$ and $p_\perp$ are used, so that $p^2 = p^+p^- - p_\perp^2$ and the propagator can be written

$$\begin{aligned} \int d^4p\ \ \delta[m_2^2 - (\tfrac{1}{2}\,P - p)^2]\ G_{LF} &= \int d^4p_2\ \delta[m_2^2 - p_2^2]\ \ [m_1^2 - (P - p_2)^2]^{-1} \\ &= \int_o^1 \frac{dx\ d^2p_\perp}{2x(1-x)}\ \ \left[\frac{m_1^2 + p_\perp^2}{1-x} + \frac{m_2^2 + p_\perp^2}{x}\ - W^2\right]^{-1} \end{aligned} \tag{6}$$

where, in the last expression, $P = (P^+, P^-, P_\perp) = (P, W^2/P, \vec{0})$ and x is the famous longitudinal momentum fraction

$$x \equiv \frac{p_2^+}{P^+} \, , \; 1-x = \frac{p_1^+}{P^+} \quad . \tag{7}$$

Note that the final answer shows that this propagator is completely symmetric under the interchange of particles 1 and 2; we would have obtained the same result if we had begun with $\delta[m_1^2 - (1/2\ P + p)^2]$. Furthermore, if both particles 1 <u>and</u> 2 are on the mass shell, then

$$p_1^- = \frac{m_1^2 + p_\perp^2}{(1-x)P^+} \; ; \; p_2^- = \frac{m_2^2 + p_\perp^2}{xP^+} \tag{8}$$

so that the LF propagator can also be written

$$\int_0^{P^+} \frac{dp^+ \, d^2p_\perp}{2\, p_2^+ \, p_1^+} \; [p_1^- + p_2^- - P^-]^{-1} \tag{9}$$

showing that the propagation is off the p^- shell. Hence, the LF formalism shows some resemblance to time ordered perturbation theory, with p^- playing the role of energy. However, the LF formalism is invariant under boosts in the z direction and rotations in the 1,2 plane, but is not invariant under rotations which mix the 3rd axis with the 1,2 plane, while the time ordered formalism is invariant under rotations, but not under boosts.

The LF formalism is very suitable for high energy problems where there is a preferred direction[8]. It has become a standard tool for the analysis of high energy quark interactions, but has seen less application to nuclear physics, where explicitly angular momentum conservation is a very useful tool[9].

It is amusing to note that the G_1 formalism and the LF formalism bear certain formal similarities[10]. If we introduce

$$x' = \frac{E_2 + p_z}{W} \tag{10}$$

we can transform (5) into (6), the only difference being that x' replaces x and the limits of integration on x' are from 0 to ∞ instead of 0 to 1.

1.3 Relationship between equations

It is important to realize that these equations are equivalent if the kernel V is chosen correctly. In proving such relations, it must be recognized that some equations depend on more variables, and equivalence can only be proved in a region where both amplitudes are defined. I will illustrate this by comparing the BS and G_1 equations. To do this introduce an operator P, which fixes the variables so that particle 2 is on shell. Suppose that M_{BS} is the BS amplitude, M_1' is its projection $M_{BS}P$, and M_1 is its double projection $PM_{BS}P$. Then if

$$M_{BS} = V_{BS} + V_{BS}\, G_{BS}\, M_{BS} \tag{11}$$

and V_1' satisfies the equation

$$V_1' = V_{BS}P + V_{BS}\,(G_{BS} - PG_1P)\, V_1' \tag{12}$$

it can be readily shown that

$$M_1' = V_1' + V_1'\, G_1\, M_1 \quad . \tag{13}$$

Hence, to determine the BS amplitude when particle 2 is on shell in either the initial <u>or</u> final state, it is sufficient to solve the G_1 equation

$$M_1 = V_1 + V_1 G_1 M_1 \tag{14}$$

where $V_1 = PV_1'$, and use Eq. (13). This theorem is well known, but usually the role of Eq. (13) is not emphasized.

The significance of these observations is that one can just as well start with the G_1 equation as with the BS equation. Since the BS kernel V_{BS} is an infinite series, use of either of these equations must inevitably involve an approximation in which this series is truncated after a finite number of terms, usually the first corresponding to one boson exchange (OBE) or one gluon exchange (OGE). I have argued elsewhere that the series for V_1 is probably more convergent than that for V_{BE}, and will not develop this further here[1,11].

1.4 Singularities

The last issue I will discuss in this section is the presence of singularities in relativistic equations. These can readily arise because of the indefinite nature of the Lorentz metric; the squared 4-momentum is not positive definite as it is in non-relativistic physics, and this leads to singularities in propagators which are not always physical.

The singularities in the BS equation are usually eliminated by performing a Wick rotation of the energy variables to complex values. For the G_1 equation, singularities occur which must be eliminated in other ways. I wish to discuss one of these, the so called dissolution singularity, and show how it is dealt with[12].

The propagator (5) can be factorized:

$$G_1(p) = (E_1 + E_2 - W)^{-1} \, (E_1 + W - E_2)^{-1} \tag{15}$$

Note that this has singularities when $W = E_1 + E_2$ and when $W = E_2 - E_1$. The former is the usual elastic cut, and is physical and should be present. The singularity at $W = E_2 - E_1$ is spurious; it would imply that the interaction is strong at small W, regardless of the dynamics.

This singularity can be removed if it is desired to use the G_1 equation for small W. To understand its origin and to see how to remove it, return to the BS propagator

$$G_{BS} = [E_1^2 - (\tfrac{W}{2} + p_o)^2 - i\epsilon]^{-1} \, [E_2^2 - (\tfrac{W}{2} - p_o)^2 - i\epsilon]^{-1} \tag{16}$$

which has four poles in the p_o complex plane, as shown in Fig. 1. When $\vec{p}$ is small and $W \simeq m_1 + m_2$, the negative energy poles are widely separated from the positive energy ones, and integrands will be dominated by either the positive energy pole of particle 1 (if the p_o integration contour is closed in the lower half plane) or by the positive energy pole of particle 2 (if the upper half plane is used). However, when W is small, the positive energy pole of particle 2 is pinched by the negative energy pole of particle 1, and a singularity will arise unless both are retained. If both are kept this would lead to a generalization of (5):

$$\int d^4p_2\ \delta_+[m_2^2 - p_2^2]\ [m_1^2 - (P - p_2)^2]^{-1} + \int d^4p_1\ \delta\ [m_1^2 - p_1^2]\ [m_2^2 - (P - p_1)^2]^{-1} \tag{17}$$

$$= \int dp_o\ d^3p \quad \frac{\delta\ (E_2 - \frac{1}{2}W + p_o)}{2\ E_2\ [E_1^2 - (W - E_2)^2]} + \frac{\delta(E_1 + \frac{1}{2}W + p_o)}{2\ E_1\ [\ E_2^2 - (W + E_1)^2]}$$

This leads to a coupled set of equations with potentials with different values of the relative energy p_o corresponding to different retardation.

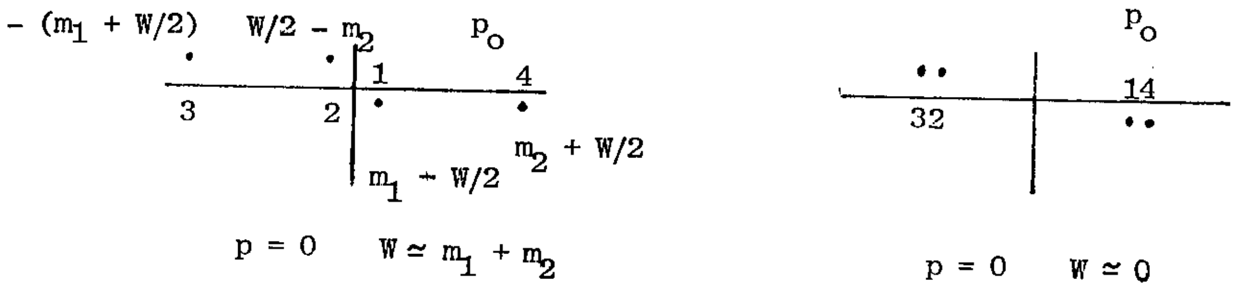

Figure 1

When $W \to E_2 - E_1$ the two δ functions become identical so that the different potentials become equal and the singularity at $W = E_2 - E_1$ cancels. At this point the propagator reduces to

$$\int dp_o\ d^3p \quad \frac{(E_1 + E_2)\ \delta(\frac{1}{2}\ (E_2 - E_1) + p_o)}{2\ E_2 E_1\ [(E_1 + E_2)^2 - W^2]} \ . \tag{18}$$

As long as one is interested in solutions far away from $W = E_2 - E_1$, it is an excellent approximation to neglect the negative energy channel. This is because the retardation factor in the potential which couples this channel to the positive energy channel is very large (making the potential small) unless W is small.

II. Relativistic Three Body Equations

Applications of relativistic two-body equations have been discussed widely in the past, and will be discussed by Tjon at this conference[13]. In the space remaining to me, I would like to discuss some recent work on a generalization of the G_1 equation to three particles[14].

The equations naturally have a Faddeev structure, with the spectator and one of the two interacting particles on shell. That the spectator must be on shell follows from a consideration of all ladder and crossed ladder exchanges between three particles. An example of a sequence of interactions between particle 1 and 2 followed by 2 and 3 is shown in Fig. 2; it can be seen in this special case that particle 1 and 3

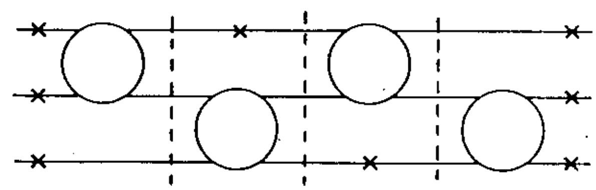

Figure 2

alternate as spectators, and the topology of the diagram requires that the propagator correspond to particles 1 and 3 on shell, and particle 2 off shell. Consideration of other interactions leads one quickly to the observation that all three propagators corresponding to the three combinations of two particles on shell are needed. The equations are shown diagrammatically in Fig. 3 in the case where all three particles are different and three body forces are neglected. Note that a symmetric treatment requires 6 Faddeev amplitudes instead of the three required in non-relativistic physics, and that the two-body driving amplitudes are identical to those obtained from the G_1 equation, except that all four two-body amplitudes corresponding to the four possible choices of which of the two particles in the initial or final state is to be on shell, are needed. For identical particles this complication disappears, since the Pauli-principle leads to the observation that the four possible two-body amplitudes are all equal, and that there is really only one Faddeev amplitude.

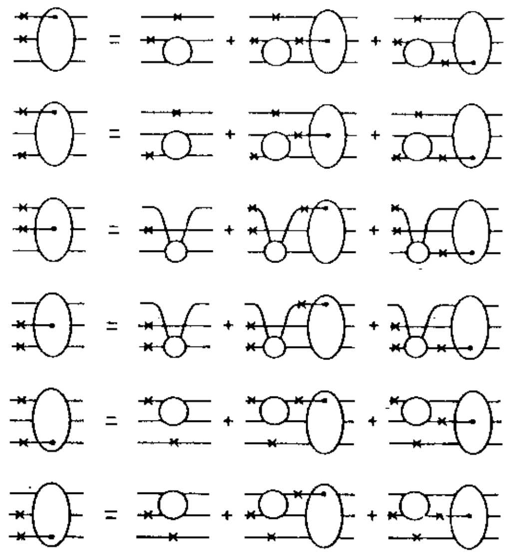

Figure 3

This approach has several advantages. First, note that the BS treatment of the relativistic 3 body problem would have 8 internal variables corresponding to the two independent 4 vector momenta. Using the G_1 equation, the internal relative energies (or relative times) are fixed in a covariant way by the two mass shell conditions, leaving two internal relative three momenta, just as in the non-relativistic case. Hence the G_1 approach has considerably simplified the problem. It is not very much harder to to solve the three body problem using the G_1 equation than it is to solve the non-relativistic three-body problem; the equations can be reduced to coupled two dimensional integral equations.

The G_1 equation also satisfies the cluster property -- in the limit when one of the three particles is removed to infinity, the interaction between the remaining two particles is independent of the presence of the third, except for the requirement of energy conservation which constrains the energy W_{12} of the pair

$$W_{12} = M_T - E_3 \tag{19}$$

where M_T is the total (CM) energy of the three body system, and E_3 is the physical, on shell energy of particle 3.

The explicit form of the bound state equation for three identical spinless particles[15] is:

$$\Gamma(p_3, \frac{1}{2}(p_1 - p_2)) = -2 \int \frac{d^3k_1}{(2\pi)^3 2E_1} \frac{M(\frac{1}{2}(p_1 - p_2), \frac{1}{2}(k_k - k_2); P - p_3)}{m_2^2 - k_2^2}$$
$$\times\ \Gamma(k_1, \frac{1}{2}(p_3 - k_2)) \tag{20}$$

where $k_2 = P - p_3 - k_1$, and $M(p,k;P_{12})$ is the two body scattering amplitude which solves Eq. (5) and $\Gamma(p_3,p)$ is the Faddeev amplitude with the spectator momentum equal to p_3 and the relative momentum of the interacting 1-2 pair equal to p. All of these quantities are manifestly covariant, so the M matrix is a world scalar.

The Lorentz invariance of the helicity formalism makes it convenient for carrying out the partial wave decomposition. Using the paper by Wick[16] the algebra can be carried out, giving the result:

$$\Gamma(P\ p\ q\ j\ m) = -\sum_{j'm'} \int \frac{dp'}{E_{p'}} \frac{q'dq'}{E_{q'}} C \frac{M^j(q, q_o; W_{12})}{W_{12}(2E_o - W_{12})} \tag{21}$$
$$\times D(\chi, \theta, \theta')\ \Gamma(P\ p'q'j'\ m')$$

where

$$C = \frac{W'_{23}((2j+1)(2j'+1))^{1/2}}{8(2\pi)^3}$$
$$D(\chi, \theta, \theta') = \theta[1 - |\cos\theta'|]\ d^J_{m'm}(\chi)\ d^j_{mo}(\theta)\ d^{j'}_{m'o}(\theta') \tag{22}$$

where $\Gamma(P, p\ q\ j\ m)$ is the projected Faddeev amplitude with p as the magnitude of the spectator 3-momentum, q the magnitude of the relative 3 momentum of the interacting pair <u>in the CM of the pair</u>, j the angular momentum of the pair (defined in its CM), and m is the projection of this angular momentum in the direction of the spectator momentum $\vec{p}$ (equal to the negative of the 3 momentum of the pair). Similarly, M^j is the j^{th} partial wave of the pair, with energy (in the CM of the pair) W_{12} and relative momenta q and q_o. The total angular momentum of the bound state is J. In terms of the variables p, q, p′ and q′ the kinematic quantities are:

$$W_{12}^2 = M_T^2 + m^2 - 2M_T E_p$$

$$W_{32}'^2 = M_T^2 + m^2 - 2M_T E_{p'}$$

$$E_o = (m^2 + q_o^2)^{1/2} = \frac{1}{2W_{12}} [W_{12}^2 - W_{23}'^2 + 2W_{23}' E_{q'}]$$

$$p'q' \cos\theta' = (M_T - E_{p'})E_{q'} - W_{23}' E_p$$

$$p\, q_o \cos\theta = (M_T - E_p)\, E_o - W_{12}\, E_{p'}$$

$$pp' \cos\chi = W_{12}\, E_o - E_{p'}(M_T - E_p) \qquad (23)$$

The d's are the usual Wigner rotation functions. The relations between p, p', q', and q_o, and the angles θ, θ', and χ are summarized in Figure 4.

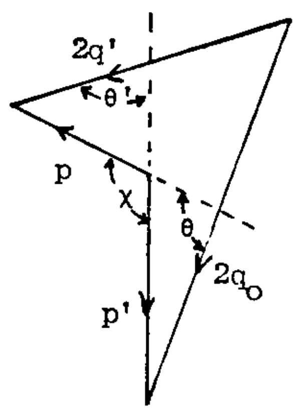

Figure 4

I have presented Eq. (21) in its entirety to show that an explicit formula exists, and to underline the fact that it is no more complicated than the usual non-relativistic formula. Eq. (21) uses the partial wave amplitudes of the two body driving terms derived in the two body rest system. The somewhat complicated relations between q_o and the integration variables p' and q' occurs naturally as a result of the covariance.

I will close this talk by noting an unusual feature of the G_1 three-body equations. As the spectator momentum increases, the energy of the interacting pair, Eq. (19), decreases. Eventually the momentum and energy become equal, which occurs at

$$p_{crit} = \frac{4}{3} m \tag{24}$$

if $M_T = 3m$. At this point the mass of the interacting pair is zero, and it is traveling at the speed of light; the Lorentz transformation effects are enormous. It looks at first as if the singularity discussed in Section 1.4 will arise here, but as it turns out the Lorentz effects over-compensate, and the amplitude is <u>zero</u> at this point, not singular! This happens because the propagator actually becomes (cf. Eq. 23)

$$W_{12}(2E_o - W_{12}) \underset{p \to p_{crit}}{\to} W_{23}'(2E_{q'} - W_{23}') \tag{25}$$

and any singularity which might arise from $W_{23}' \to 0$ is cancelled by a similar factor in the numerator. Hence the propagator is finite, but because $q_o \to \infty$, the amplitudes $M^j \to 0$, making $\Gamma = 0$ at the critical point. Hence the 3 body equations can be truncated to $p \le p_{crit}$; they are zero at the boundary and above this region are driven by two body amplitudes with space-like 4-momenta, a region of small effects safely ignorable.

The fully relativistic three body problem is therefore ready to be solved, but given experience with the non-relativistic problem we can anticipate that good numerical results are not likely to be ready for several years!

Acknowledgments

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References

1. F. Gross in Nuc. Phys. A416 (1984) 387C; in "Electron and Photon Interations at Intermediate Energies", Ed. by D. Menze et. al., Springer lecture notes in physics #234, p. 292, in "Few-Body Problems in Physics", Ed. by L.D. Faddeev and T.I. Kopaleishvili, World Scientific, p. 344; in "Proceedings of the LAMPF Workshop on Dirac Approaches to Nuclear Physics; LA-10438-C, p. 351; in the

"Second Workshop on Prospectives in Nuclear Physics at Intermediate Energies", Trieste, March 1985 Ed. by S. Boffi, C. Ciofi degli Atti, and M.M. Giannini, World Scientific, p. 72.

2. F.P. Juster, et. al., Phys. Rev. Lett. 55 (1985) 2261.

3. W. Turchinetz, Proceedings of the CEBAF 1986 Summer Workshop, to be published.

4. See K. Holinde and R. Machleidt, Nucl. Phys. A327 (1981) 349; R. Machleidt, Trieste Workshop op. crit. p. 282.

5. See E.F. Van Faassen and J.A. Tjon, Phys. Rev. C24 (1981) 736; 28 (1983) 234; 30 (1984) 285; 33 (1986) 2105. J.A. Tjon to be published in the proceedings of the Tokyo Few Body Conference, August 1986, and contributions to this conference.

6. See F. Gross, Phys. Rev. 186 (1969) 1448; D10 (1974) 233.

7. See J. M. Namyslowski, Proceedings of the Graz Conference (1978) Lecture Notes in Physics #82, Ed. by Zingl et. al., p. 41; M. Chemtob, Nuc. Phys. A336 (1980) 299; L.L. Frankfurt and M.I. Strickman, Nucl. Phys. B148 (1979) 107; Physics Reports 76 (1981) 215.

8. G.P. Lepage and S.J. Brodsky, Phys. Rev. D22 (1980) 2157.

9. See, however, work by Frankfurt and Strickman, ref. 7.

10. F. Gross and B.D. Keister, Phys. Rev. C28, (1983) 823.

11. F. Gross, Phys. Rev. C26 (1982) 2203.

12. I am indebted to S.J. Wallace for calling my attention to this issue and for helpful discussions. See also V.B. Mandelzweig and S.J. Wallace, "QED Based Two-Body Dirac Equation", submitted for publication.

13. J. Tjon, contribution to this conference.

14. F. Gross, Phys. Rev. C26 (1982) 2226.

15. F. Gross, notes for lectures given at the Universitat Hannover, 1983, unpublished.

16. Wick, Ann. of Phys. (NY) 18 (1962) 65.